The Fundamental Group of the Circle

Fundamental Groups Covering Maps

Exercise: Visualize a homotopy

Is this desmos demo a homotopy?

Fundamental Groups of Familiar Spaces

The tools we use:

Definition: The **lifting correspondence**

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is defined by $\phi([f]) = \tilde{f}(1)$, where $\tilde{f}$ is the unique lift of f starting at e_0.

The the lifting correspondence is a surjection and a bijection if E is simply connected.

Thus the set of homotopy classes of loops in B based at b_0 counts the number of sheets in the cover $p : E \to B$ over b_0.

Circle

$\pi_1(S^1) = \mathbb{Z}$.

Every loop in the circle is essentially just "winding" around the center n times.

Torus

The torus has $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$.

$$\pi_1(T) =< a, b|aba^{-1}b^{-1} >$$

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This comes from the cell-structure on the torus where the 1-cells (edges) give the generators and the 2-cell (face) the relation (paths along the edges can be homotopied across the face).

This forces $ab = ba$ which forces the group to be ablian. The abelian group with two generators is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

The torus has an annulus ($\pi_1(A) \cong \mathbb{Z}$) as intermediate infinite cover. The lifting correspondence is a surjection.

Figure-8

The two-petal rose has the *free-group* on two-generators as fundamental group. An intermediate cover is the infinite string of circles. The lifting-correspondence

$$\pi_1 :< a, b >\to p^{-1}(x_0)$$

is surjective as $p^{-1}(x_0) \cong \mathbb{Z}$.

Impossibility Proofs

Functoriality allows us to translate topological problems into algebraic ones. If a certain map cannot exist in the "Group world," the corresponding map cannot exist in the "Space world."

Definition: A **retraction** of a space X onto a subspace A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. In other words, the restriction of r to A is the identity map on A.

> **No-Retraction Lemma** There is no continuous map $r : D^n \to \partial D^n$ such that $r(x) = x$ for all $x \in \partial D^n$.

Proof: If there were a retraction $r : D^2 \to S^1$, then by functoriality, the following diagram must commute:

$$\begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{i_*} & \pi_1(D^2) & \xrightarrow{r_*} & \pi_1(S^1) \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\ n & \mapsto & 0 & \mapsto & 0 \end{array}$$

The Contradiction: Since $r \circ i = id_{S^1}$, functoriality requires $(r \circ i)_* = id_{\mathbb{Z}}$. But as shown above, the composition $r_* \circ i_*$ maps everything to 0. Thus, no such retraction r can exist.

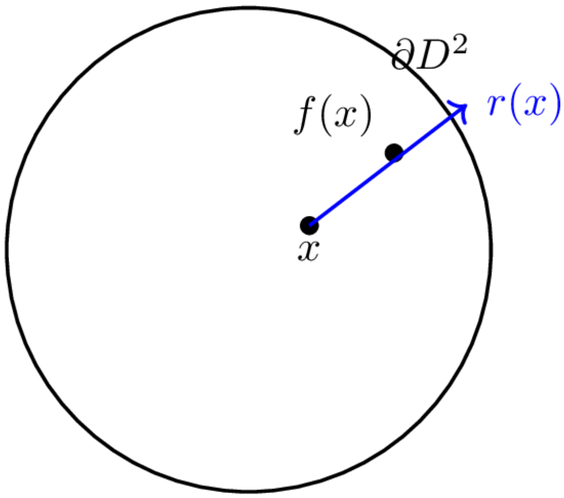

Figure 1: Hypothetical retraction of the ball onto its boundary

Brouwer Fixed Point Theorem

A fundamental result in topology. In this proof, we demonstrate that the existence of a fixed-point-free map would contradict the **No-Retraction Lemma**.

> **Brouwer Fixed Point Theorem** Let D^n be the closed unit n-disk. Any continuous map $f : D^n \to D^n$ has at least one fixed point; that is, a point $x \in D^n$ such that $f(x) = x$.

Proof Sketch

We proceed by contradiction. Assume there exists a continuous map $f : D^n \to D^n$ such that $f(x) \neq x$ for all $x \in D^n$.

1. Construction of the Retraction

Since $f(x) \neq x$, these two points determine a unique ray. Define $r : D^n \to \partial D^n$ by sending x to the point where the ray starting at $f(x)$ and passing through x hits the boundary ∂D^n.

Specifically, we look for $r(x) = x + t(f(x) - x)$ with $t \geq 0$ such that $\|r(x)\| = 1$. See the Desmos demo: link

2. Continuity

Because f is continuous and $f(x)$ never coincides with x, the vector $f(x) - x$ is never zero and varies continuously. Solving for the positive root of the resulting quadratic equation in t shows that $r(x)$ is a continuous function.

3. Boundary Conditions

If $x \in \partial D^n$, then $\|x\| = 1$. The ray starting at $f(x)$ and passing through x is already at the boundary when it reaches x. Thus, $r(x) = x$ for all x on the boundary.

4. Conclusion

The map $r : D^n \to \partial D^n$ is a continuous retraction of the disk onto its boundary. This directly contradicts the **No-Retraction Lemma**. Therefore, our initial assumption was false: f must have at least one fixed point.

The image below is for a nice Mobius transformation of the disk to itself.

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

Keep in mind that the theorem applies to any continuous endomorphism of the disk, no matter how wild.

A related result is

> **Borsuk-Ulam Theorem:** You can't map S^2 into $\mathbb{R}^2$ without collapsing two antipodal points.

Proof: The no-retraction theorem above generalizes to n-dimensions, and the preceeding proof applies with little modification.

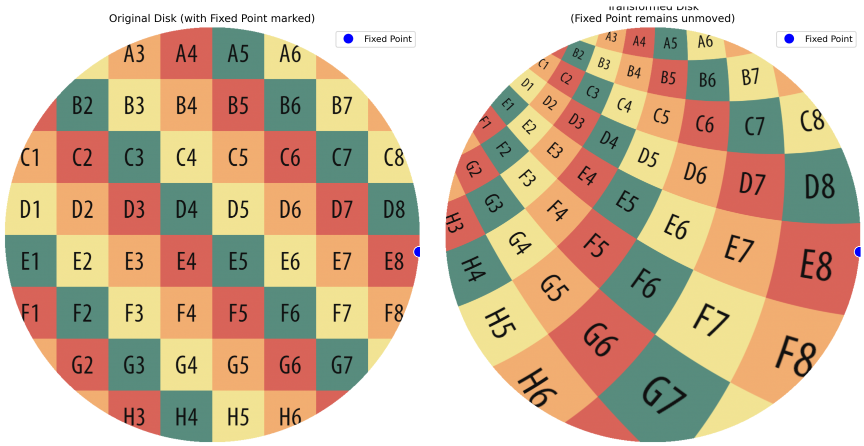

Figure 2:

Deformation Retracts & Homotopy Types

> A **homotopy equivalence** is a continuous map $f : X \to Y$ such that there exists a continuous map $g : Y \to X$ with $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. In this case, we say that X and Y have the same **homotopy type**.

Intuitively, a homotopy equivalence is a map that has a "homotopy inverse," meaning that it can be "undone" up to homotopy. If two spaces are homotopy equivalent, they are considered to be the same from the perspective of homotopy theory, as they have the same fundamental group and other homotopy invariants.

One important consequence of homotopy equivalence is that it induces an isomorphism on π_1:

Theoreme: If $f : X \to Y$ is a homotopy equivalence, then the induced map on fundamental groups $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

One example of a homotopy equivalence is the inclusion of the circle into the punctured plane:

Theorem: The inclusion $j : S^1 \to \mathbb{R}^2 - \{0\}$ is a homotopy equivalence. In particular, $\pi_1(\mathbb{R}^2 - \{0\}) \simeq \pi_1(S^1) \simeq \mathbb{Z}$.

A related (actually, less general) notion is that of a deformation retract:

Deformation Retract: A subspace A of a space X is a **deformation retract** if there exists a homotopy $H : X \times [0, 1] \to X$ such that:

1. $H(x, 0) = x$ for all $x \in X$ (the homotopy starts at the identity),

2. $H(x,1) \in A$ for all $x \in X$ (the homotopy ends in A),
3. $H(a,t) = a$ for all $a \in A$ and $t \in [0,1]$ (points in A remain fixed throughout the homotopy).

Intuitively, a deformation retract is a way to "shrink" a space X down to a subspace A without tearing or gluing, and while keeping the points of A fixed.

Theorem: If A is a deformation retract of X, then the inclusion map $i : A \to X$ is a homotopy equivalence, and thus $\pi_1(A) \cong \pi_1(X)$

Finally, just to relate the notions of homotopy equivalence and deformation retract, we note the following result:

Theorem: If $f : X \to Y$ is a homotopy equivalence, then $\exists$ a space Z such that X and Y are both deformation retracts of Z.

The fundamental group of some surfaces

Sphere: $\pi_1(S^2) = \{0\}$

Torus: $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$

Projective Plane: $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$

Double Torus: $\pi_1(\Sigma_2) =< a, b, c, d | aba^{-1}b^{-1}cdc^{-1}d^{-1} >$ (the free group on 4 generators modulo the relation that the product of the commutators is the identity).

Mobius Strip: $\pi_1(M) = \mathbb{Z}$