The Fundamental Group of the Circle

Fundamental Groups Covering Maps

Fundamental Groups

Circle

$\pi_1(S^1) = \mathbb{Z}$.

Every loop in the circle is essentially just “winding” around the center n times.

Torus

The torus has $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$.

$$\pi_1(T) =< a, b | aba^{-1}b^{-1} >$$

.

This comes from the cell-structure on the torus where the 1-cells (edges) give the generators and the 2-cell (face) the relation (paths along the edges can be homotopied across the face).

This forces $ab = ba$ which forces the group to be ablian. The abelian group with two generators is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Figure-8

The two-petal rose has the *free-group* on two-generators as fundamental group. An intermediate cover is the infinite string of circles. The lifting-correspondence

$$\pi_1 :< a, b > \to p^{-1}(x_0)$$

is surjective as $p^{-1}(x_0) \cong \mathbb{Z}$.

Impossibility Proofs

Functoriality allows us to translate topological problems into algebraic ones. If a certain map cannot exist in the "Group world," the corresponding map cannot exist in the "Space world."

Definition: A **retraction** of a space X onto a subspace A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. In other words, the restriction of r to A is the identity map on A.

> **No-Retraction Lemma** There is no continuous map $r : D^n \to \partial D^n$ such that $r(x) = x$ for all $x \in \partial D^n$.

Proof: If there were a retraction $r : D^2 \to S^1$, then by functoriality, the following diagram must commute:

$$\begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{i_*} & \pi_1(D^2) & \xrightarrow{r_*} & \pi_1(S^1) \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\ n & \mapsto & 0 & \mapsto & 0 \end{array}$$

The Contradiction: Since $r \circ i = id_{S^1}$, functoriality requires $(r \circ i)_* = id_{\mathbb{Z}}$. But as shown above, the composition $r_* \circ i_*$ maps everything to 0. Thus, no such retraction r can exist.

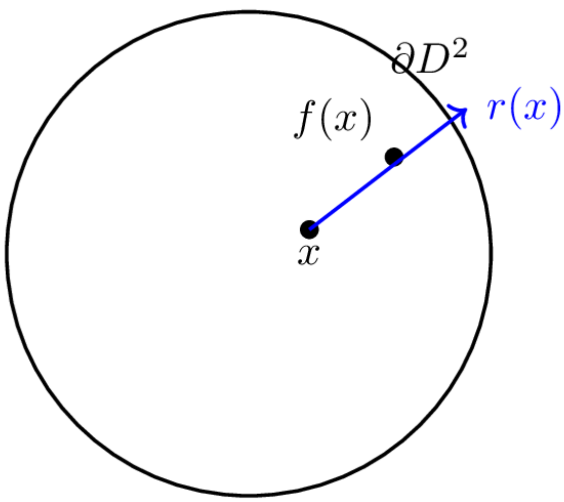


Figure 1: Hypothetical retraction of the ball onto its boundary

Brouwer Fixed Point Theorem

A fundamental result in topology. In this proof, we demonstrate that the existence of a fixed-point-free map would contradict the **No-Retraction Lemma**.

> **Brouwer Fixed Point Theorem** Let D^n be the closed unit n-disk. Any continuous map $f : D^n \to D^n$ has at least one fixed point; that is, a point $x \in D^n$ such that $f(x) = x$.

Proof Sketch

We proceed by contradiction. Assume there exists a continuous map $f : D^n \to D^n$ such that $f(x) \neq x$ for all $x \in D^n$.

1. Construction of the Retraction

Since $f(x) \neq x$, these two points determine a unique ray. Define $r : D^n \to \partial D^n$ by sending x to the point where the ray starting at $f(x)$ and passing through x hits the boundary ∂D^n.

Specifically, we look for $r(x) = x + t(f(x) - x)$ with $t \geq 0$ such that $\|r(x)\| = 1$. See the Desmos demo: link

2. Continuity

Because f is continuous and $f(x)$ never coincides with x, the vector $f(x) - x$ is never zero and varies continuously. Solving for the positive root of the resulting quadratic equation in t shows that $r(x)$ is a continuous function.

3. Boundary Conditions

If $x \in \partial D^n$, then $\|x\| = 1$. The ray starting at $f(x)$ and passing through x is already at the boundary when it reaches x. Thus, $r(x) = x$ for all x on the boundary.

4. Conclusion

The map $r : D^n \to \partial D^n$ is a continuous retraction of the disk onto its boundary. This directly contradicts the **No-Retraction Lemma**. Therefore, our initial assumption was false: f must have at least one fixed point.

The image below is for a nice Mobius transformation of the disk to itself.

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

Keep in mind that the theorem applies to any continuous endomorphism of the disk, no matter how wild.

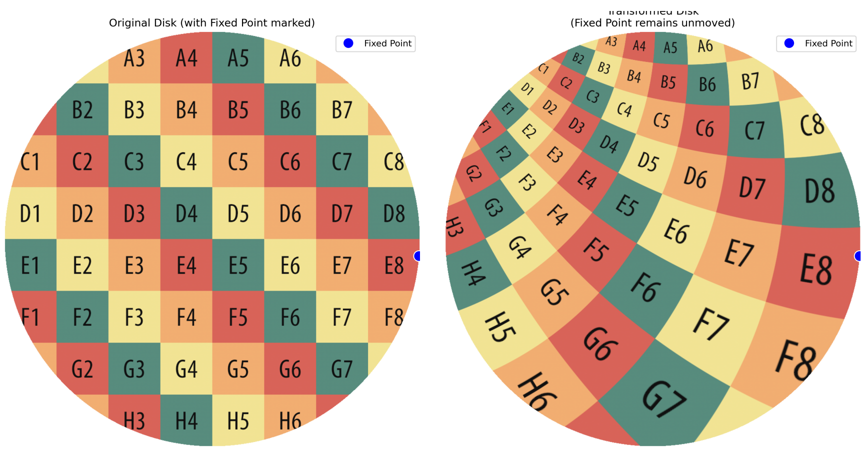


Figure 2:

A related result is

> **Borsuk-Ulam Theorem:** You can't map S^2 into $\mathbb{R}^2$ without collapsing two antipodal points.

Proof: The no-retraction theorem above generalizes to n-dimensions, and the preceeding proof applies with little modification.

Deformation Retracts & Homotopy Types

> A **homotopy equivalence** is a continuous map $f : X \to Y$ such that there exists a continuous map $g : Y \to X$ with $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. In this case, we say that X and Y have the same **homotopy type**.

Intuitively, a homotopy equivalence is a map that has a "homotopy inverse," meaning that it can be "undone" up to homotopy. If two spaces are homotopy equivalent, they are considered to be the same from the perspective of homotopy theory, as they have the same fundamental group and other homotopy invariants.

One important consequence of homotopy equivalence is that it induces an isomorphism on π_1:

Theoreme: If $f : X \to Y$ is a homotopy equivalence, then the induced map on fundamental groups $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

One example of a homotopy equivalence is the inclusion of the circle into the punctured plane:

Theorem: The inclusion $j : S^1 \to \mathbb{R}^2 - \{0\}$ is a homotopy equivalence. In particular, $\pi_1(\mathbb{R}^2 - \{0\}) \simeq \pi_1(S^1) \simeq \mathbb{Z}$.

A related (actually, less general) notion is that of a deformation retract:

Deformation Retract: A subspace A of a space X is a **deformation retract** if there exists a homotopy $H : X \times [0, 1] \to X$ such that:

1. $H(x, 0) = x$ for all $x \in X$ (the homotopy starts at the identity),
2. $H(x, 1) \in A$ for all $x \in X$ (the homotopy ends in A),
3. $H(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$ (points in A remain fixed throughout the homotopy).

Intuitively, a deformation retract is a way to "shrink" a space X down to a subspace A without tearing or gluing, and while keeping the points of A fixed.

Theorem: If A is a deformation retract of X, then the inclusion map $i : A \to X$ is a homotopy equivalence, and thus $\pi_1(A) \cong \pi_1(X)$

Finally, just to relate the notions of homotopy equivalence and deformation retract, we note the following result:

Theorem: If $f : X \to Y$ is a homotopy equivalence, then $\exists$ a space Z such that X and Y are both deformation retracts of Z.

The fundamental group of some surfaces

Sphere: $\pi_1(S^2) = \{0\}$

Torus: $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$

Projective Plane: $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$

Double Torus: $\pi_1(\Sigma_2) =< a, b, c, d | aba^{-1}b^{-1}cdc^{-1}d^{-1} >$ (the free group on 4 generators modulo the relation that the product of the commutators is the identity).

Mobius Strip: $\pi_1(M) = \mathbb{Z}$

The Siefert Van Kampen Theorem

Fundamental groups are often difficult to compute directly from the definition. The Siefert Van Kampen theorem allows us to compute the fundamental group of a space X by breaking it up into simpler pieces.

First imagine our space X is the union of two open sets U and V such that $U \cap V$ is path-connected. Then we can use the Siefert Van Kampen theorem to compute $\pi_1(X)$ from $\pi_1(U)$, $\pi_1(V)$, and $\pi_1(U \cap V)$.

First we state a theorem that states that, given any loop f in X based at x_0, it is path homotopic to a loop that is of the form

$$g_1 * (g_2 * (\cdots * (g_{n-1} * g_n) \dots))$$

where each g_i is a loop in U or V. In other words, we can "break up" any loop into pieces that lie in the U,V pieces.

Then we can use this to show that $\pi_1(X)$ is generated by the images of $\pi_1(U)$ and $\pi_1(V)$ under the inclusion maps, and that the only relations are those coming from the intersections $U_\alpha \cap U_\beta$.

Theorem: Let $X = U \cup V$ where U and V are open subsets of X such that $U \cap V$ is path-connected and that $x_0 \in U \cap V$. Let i and j be the inclusion maps of $U \cap V$ into U and V respectively. Then $\pi_1(X)$ is generated by the images of $\pi_1(U)$ and $\pi_1(V)$ under the inclusion maps, and the only relations are those coming from the intersections $U \cap V$.

Corollary: If $U \cap V$ is simply connected, then $\pi_1(X) \cong \pi_1(U) * \pi_1(V)$ (the free product of the two groups).

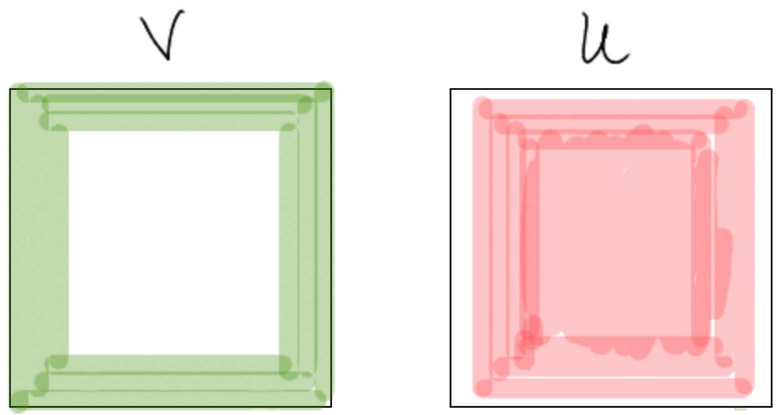
Example: The figure-8 space can be decomposed into two circles U and V that intersect at a single point. Since the intersection is simply connected, we have $\pi_1(\text{figure-8}) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}$ (the free group on two generators).

The Fundamental Group of the Torus via Seifert-van Kampen

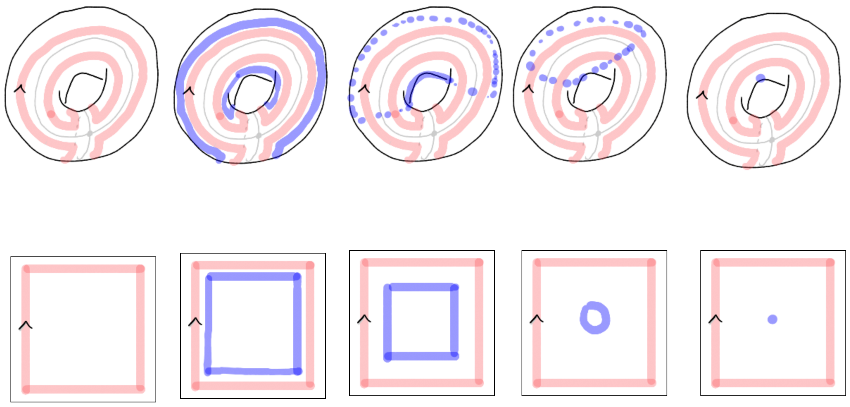
To compute $\pi_1(T^2)$, we view the torus as a quotient of a square I^2 with edges identified according to the word $aba^{-1}b^{-1}$.

1. The Decomposition

We decompose T^2 into two open sets, U and V:



- U**:** An open disk representing the "interior" of the square. U is contractible.
  - $\pi_1(U) \cong \{1\}$
- V**:** The 1-skeleton (the edges a and b) plus a small neighborhood extending into the interior. V deformation retracts onto $S^1 \vee S^1$.
  - $\pi_1(V) \cong \langle a, b \mid \emptyset \rangle \cong \mathbb{Z} * \mathbb{Z}$
- $U \cap V$**:** An open "annular" strip that deformation retracts onto a circle.
  - $\pi_1(U \cap V) \cong \langle \gamma \mid \emptyset \rangle \cong \mathbb{Z}$



2. The Amalgamated Free Product

The Seifert-van Kampen Theorem identifies the fundamental group of the union as the amalgamated free product:

$$\pi_1(T^2) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

We look at the induced maps from the fundamental group of the intersection:

1. **Map** $i_* : \pi_1(U \cap V) \to \pi_1(U)$**:** Since U is contractible, $i_*(\gamma) = 1$.
2. **Map** $j_* : \pi_1(U \cap V) \to \pi_1(V)$**:** The loop γ travels along the boundary of the square. Following the identification, this path is $a \to b \to a^{-1} \to b^{-1}$.
   - $j_*(\gamma) = aba^{-1}b^{-1}$

3. Calculation

The presentation for the amalgamated free product is:

$$\pi_1(T^2) \cong \langle a, b \mid i_*(\gamma) = j_*(\gamma) \rangle$$

Substituting our specific maps:

$$\pi_1(T^2) \cong \langle a, b \mid 1 = aba^{-1}b^{-1} \rangle$$

Result

The relation $1 = aba^{-1}b^{-1}$ is equivalent to $ab = ba$. Thus, the fundamental group of the torus is the free abelian group on two generators:

$$\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$$

The Fundamental Group of the Klein Bottle

To compute $\pi_1(K)$, we view the Klein bottle as a quotient of a square I^2 with edges identified according to the word $abab^{-1}$.

1. The Decomposition

Similar to the torus, we decompose K into two open sets U and V:

- **U:** The interior of the square (an open disk). U is contractible.
  - $\pi_1(U) \cong \{1\}$
- **V:** The 1-skeleton (the edges a and b) plus a neighborhood. V deformation retracts onto $S^1 \vee S^1$.
  - $\pi_1(V) \cong \langle a, b \mid \emptyset \rangle$
- **$U \cap V$:** An open strip that deformation retracts onto a circle γ.
  - $\pi_1(U \cap V) \cong \mathbb{Z}$

2. The Amalgamated Free Product

We apply the Seifert-van Kampen Theorem:

$$\pi_1(K) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

1. **Map $i_* : \pi_1(U \cap V) \to \pi_1(U)$:** As U is contractible, $i_*(\gamma) = 1$.
2. **Map $j_* : \pi_1(U \cap V) \to \pi_1(V)$:** Following the boundary of the square for the Klein bottle, the path is a followed by b, then a again, then b^{-1}.
   - $j_*(\gamma) = abab^{-1}$

3. Calculation

The presentation for the amalgamated free product is:

$$\pi_1(K) \cong \langle a, b \mid 1 = abab^{-1} \rangle$$

❗ Comparison with the Torus

Unlike the torus, where the relation $aba^{-1}b^{-1} = 1$ leads to an abelian group ($\mathbb{Z}^2$), the Klein bottle's relation is:

$$aba = b$$

This group is **non-abelian**. It can be viewed as a semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}$, reflecting the twist in the surface's topology.