Classification of Surfaces

n-manifolds

A 2-manifold (or surface) is a space locally homeomorphic to $\mathbb{R}^2$. Today we discuss the classification of 2-manifolds.

> **Research Ideas**
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> Manifolds, by definition, are a subject of topology. Research is focused on *classification* of manifolds, which is the problem of determining when two manifolds are homeomorphic. In this lecture, we will discuss the classification of compact connected 2-manifolds. The classification of 3-manifolds is much more complex and was only recently completed by Perelman. **A survey of the 3-manifold situation would be a nice thesis project.** There are still open questions about the classification of 4-manifolds.
> Current research in 2-manifolds involves examining the relationship between the topology of a surface and its geometry. For example, the Gauss-Bonnet theorem relates the total curvature of a surface to its Euler characteristic, which is a topological invariant. Another area of research is the study of moduli spaces of surfaces, which are spaces that parametrize all possible complex structures on a given topological surface.

Every connected, compact 2-manifold is uniquely determined by three properties:

1. **Orientability** (Orientable or Non-orientable)
2. **Genus** (g or k)
3. **Boundary Components** (n)

1. Closed Orientable Surfaces

These are surfaces with no boundary that have distinct "sides."

- **Sphere (S^2):** Genus $g = 0$.
- **Torus (T^2):** Genus $g = 1$. Shaped like a donut.
- **Genus g Surface:** A sphere with g handles.

> **Note**
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> A surface with genus g is the connected sum of g tori.

2. Closed Non-Orientable Surfaces

These surfaces contain a Möbius band and have no "inside" vs "outside."

- **Projective Plane ($\mathbb{R}P^2$):** A sphere with 1 cross-cap.
- **Klein Bottle (K):** A sphere with 2 cross-caps.
- **Connected Sum of k Projective Planes:** A sphere with k cross-caps.

3. Provisions for Boundary

Any surface with boundary is created by removing b open disks from a closed surface.

- **Disk:** Sphere with $b = 1$.
- **Annulus (Cylinder):** Sphere with $b = 2$.
- **Möbius Strip:** Projective plane with $b = 1$.
- **Pairs of Pants:** Sphere with $b = 3$.

Summary of Classification

Surface	Genus	Boundary (n)	Orientable?
Sphere	0	0	Yes
Torus	1	0	Yes
Projective Plane	1	0	No
Disk	0	1	Yes
Punctured Torus	1	1	Yes
Möbius Strip	1	1	No
Genus-2 Surface	2	0	Yes
Klein Bottle	2	0	No

The Classification Theorem

Any compact connected surface is homeomorphic to:

- A sphere ($g = 0, n = 0$)
- A connected sum of g tori (with n disks removed)
- A connected sum of k projective planes (with n disks removed)

Relationship with Euler Characteristic

The Euler characteristic χ of a surface can be calculated using the formula:

$$\chi = 2 - 2g - n \quad \text{(for orientable surfaces)}$$

$$\chi = 2 - k - n \quad \text{(for non-orientable surfaces)}$$

This invariant helps in distinguishing between different types of surfaces.

Relationship with Fundamental Group

The fundamental group π_1 of a surface also provides insight into its structure:

- For a genus g orientable surface, π_1 is generated by $2g$ loops.
- For a non-orientable surface with k projective planes, π_1 is generated by k loops with specific relations.

> **Note**
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> The problem of determining whether two group presentations are isomorphic is undecidable, which is why the classification of surfaces is so important. It provides a way to determine when two surfaces are homeomorphic without having to solve the isomorphism problem for their fundamental groups.

In fact, for arbitrary groups, there does not exist a computer algorithm that takes two finite group presentations and decides whether or not the groups are isomorphic, regardless of how (finitely) much time is allowed for the algorithm to run and how (finitely) much memory is available. However,even though the isomorphism problem for groups is undecidable in general, it is decidable for the fundamental groups of surfaces. This is because the fundamental groups of surfaces have a very specific structure that allows us to determine their isomorphism class. The classification of surfaces provides a way to determine when two surfaces are homeomorphic without having to solve the isomorphism problem for their fundamental groups.