# Basis for a topology

## Basis for a Topology

There are too many subsets to describe a topology by listing all its open sets. Instead, we can describe a topology by specifying a basis. A basis is a collection of open sets such that every open set in the topology can be expressed as a union of basis elements.

### Definition

For a set $X$, a collection $\mathcal{B}$ of subsets of $X$ is called a **basis** for a topology on $X$ if:

1. For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ such that $x \in B$.
2. If $x$ belongs to the intersection of two basis elements $B_1$ and $B_2$, then there is a basis element $B_3$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Whenever a collection $\mathcal{B}$ of subsets of $X$ satisfies these two conditions, we can define a topology $\mathcal{T}$ on $X$ by declaring a

set $U \subseteq X$ to be **open** (i.e., $U \in \mathcal{T}$) if and only if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

**Note**: Basis elements themselves are open sets in the topology generated by the basis (just replace $U$ with some basis element $B$ in the previous statement.

### Examples

1. **Standard Topology on** $\mathbb{R}$: The collection of all open intervals $(a, b)$ where $a < b$ forms a basis for the standard topology on the real numbers $\mathbb{R}$.

2. **Discrete Topology**: The collection of all singletons $\{x\}$ for each $x \in X$ forms a basis for the discrete topology on $X$, where every subset of $X$ is open.

3. **Lower Limit Topology**: The collection of all half-open intervals $[a, b)$ where $a < b$ forms a basis for the lower limit topology on $\mathbb{R}$.

4. $\mathcal{B}^\circ$ = {all open balls in $(\mathbb{R}^2$, d)} is a basis for the **standard topology on** $\mathbb{R}^2$.
5. $\mathcal{B}_1$ = {all open squares in $(\mathbb{R}^2$, d)} is a basis for the **standard topology on** $\mathbb{R}^2$. (Thus, a topology can have multiple different bases.)

### A Basis Really Does Define a Topology

1. Since the null set contains nothing it satisfies the definition of open set vacuously, so $\phi$ is open. Also, for each $x \in X$, by condition 1 of the basis definition, there is a basis element $B$ such that $x \in B$. Since $B \subseteq X$, it follows that $X$ is open. Thus, $\phi$ and $X$ are in the topology defined by the basis.
2. Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets in the topology defined by the basis $\mathcal{B}$. We want to show that $U = \bigcup_{\alpha \in A} U_\alpha$ is also open. Let $x \in U$. Then $x \in U_{\alpha_0}$ for some $\alpha_0 \in A$. (This is by defintion of union) Since $U_{\alpha_0}$ is open, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U_{\alpha_0}$. But since $U_{\alpha_0} \subseteq U$, it follows that $B \subseteq U$. Thus, for each $x \in U$, there is a basis element $B$ such that $x \in B$ and $B \subseteq U$. Therefore, $U$ is open.
3. To show that finite intersections of open sets are open we use induction. For the base case, let $U_1$ and $U_2$ be open sets in the topology defined by the basis $\mathcal{B}$. We want to show that $U = U_1 \cap U_2$ is also open. Let $x \in U$. Then $x \in U_1$ and $x \in U_2$. Since $U_1$ is open, there is a basis element $B_1 \in \mathcal{B}$ such that $x \in B_1$ and $B_1 \subseteq U_1$. Similarly, since $U_2$ is open, there is a basis element $B_2 \in \mathcal{B}$ such that $x \in B_2$ and $B_2 \subseteq U_2$.

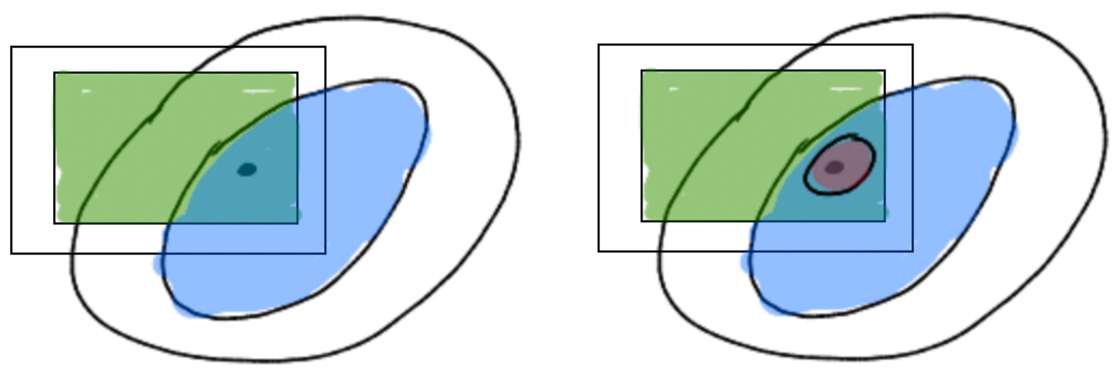

Figure 1: Shaded sets are basis elements. The nesting condition implies there must be another basis element withing their intersection.

Now, since $x \in B_1 \cap B_2$, by condition 2 of the basis definition, there is a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. But since $B_1 \subseteq U_1$ and $B_2 \subseteq U_2$, it follows that $B_3 \subseteq U_1 \cap U_2 = U$. Thus, for each $x \in U$, there is a basis element $B_3$ such that $x \in B_3$ and $B_3 \subseteq U$. Therefore,

$U$ is open. To proceed by induction one would assume that the intersection of $n$ open sets is open and then show that the intersection of $n+1$ open sets is open. The intersection plays the role of $U_1$ in the base case, and the argument proceeds similarly.

### Union Lemma: A topology is the union of its basis elements

**Proof:** We show $\tau \subseteq \bigcup_{B \in \mathcal{B}} B$ and $\bigcup_{B \in \mathcal{B}} B \subseteq \tau$.

1. Let $U \in \tau$. For each $x \in U$, there is a basis element $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subseteq U$. Thus,
$$U = \bigcup_{x \in U} B_x$$
and so $U$ is a union of basis elements. Therefore, $U \in \bigcup_{B \in \mathcal{B}} B$.
2. Now let $U \in \bigcup_{B \in \mathcal{B}} B$. Since each basis element is open, it follows that each $B \in \tau$. Since $\tau$ is a topology, it is closed under arbitrary unions, so $U \in \tau$.

### Example

Unions of open circles in $\mathbb{R}^2$ give the same topology as unions of open squares in $\mathbb{R}^2$.

### Lemma: How to find a basis for a topology

If $\mathcal{C}$ is a collection of open sets in a topology $\tau$ on $X$ such that for each open set $U \in \tau$ and each $x \in U$, there is a set $C \in \mathcal{C}$ such that $x \in C \subseteq U$, then $\mathcal{C}$ is a basis for $\tau$.

**Proof:** We need to show that $\mathcal{C}$ satisfies the two conditions of the basis definition.

1. For each $x \in X$, since $X$ is open, by assumption there is an open set $C \in \mathcal{C}$ such that $x \in C \subseteq X$. Thus, condition 1 is satisfied.
2. Assume $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$. Since $C_1$ and $C_2$ are open sets in $\tau$, their intersection $C_1 \cap C_2$ is also an open set in $\tau$. By assumption, there is a set $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Thus, condition 2 is satisfied.

There is one more point. The topology generated by $\mathcal{C}$ is $\tau$ itself. To see this, let $U$ be an open set in the topology generated by $\mathcal{C}$. Then for each $x \in U$, there is a set $C \in \mathcal{C}$ such that $x \in C$ and $C \subseteq U$. Since each $C$ is open in $\tau$, it follows that $U$ is a union of open sets in $\tau$, and hence $U$ is open in $\tau$. Conversely, let $U$ be an open set in $\tau$. By assumption, for each $x \in U$, there is a set $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Thus, $U$ is open in the topology generated by $\mathcal{C}$.

## Coarse and Fine Topologies

Given a set $X$, if $\tau_1$ and $\tau_2$ are two topologies on $X$, we say that $\tau_1$ is **coarser** than $\tau_2$ (or that $\tau_2$ is **finer** than $\tau_1$) if $\tau_1 \subseteq \tau_2$. In other words, every open set in $\tau_1$ is also an open set in $\tau_2$.

## Lemma: Comparing topologies via their bases

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\tau$ and $\tau'$ on a set $X$. Then $\tau$ is coarser than $\tau'$ ($\tau'$ is finer than $\tau$) if and only if for each basis element $B \in \mathcal{B}$ and each $x \in B$, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

**Proof:**

1. Assume $\tau$ is coarser than $\tau'$. Let $B \in \mathcal{B}$ and let $x \in B$. Since $B$ is open in $\tau$ and $\tau \subseteq \tau'$, it follows that $B$ is also open in $\tau'$. By the basis definition for $\mathcal{B}'$, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B'$ and $B' \subseteq B$.

2. Now assume that for each basis element $B \in \mathcal{B}$ and each $x \in B$, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Let $U$ be an open set in $\tau$. We want to show that $U$ is also open in $\tau'$. Let $x \in U$. Since $U$ is open in $\tau$, by the basis definition for $\mathcal{B}$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. By our assumption, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B'$ and $B' \subseteq B$. Since $B \subseteq U$, it follows that $B' \subseteq U$. Thus, for each $x \in U$, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B'$ and $B' \subseteq U$. Therefore, $U$ is open in $\tau'$. Hence, $\tau'$ is finer than $\tau$.

## Examples

1. The standard topology on $\mathbb{R}$ is finer than the lower limit topology on $\mathbb{R}$ since the lower limit basis elements $[a, b)$ can be expressed as unions of standard basis elements $(a, b)$.

2. The discrete topology on a set $X$ is finer than any other topology on $X$ since every subset of $X$ is open in the discrete topology.

3. The standard topology on $\mathbb{R}^2$ is finer than the topology generated by open squares since for any open square and any point within it, we can find an open ball (basis element of the standard topology) that fits inside the square. Conversely, the topology generated by open squares is finer than the standard topology on $\mathbb{R}^2$ since for any open ball and any point within it, we can find an open square (basis element of the square topology) that fits inside the ball. Thus, these two topologies are actually the same.

4. The **finite complement topology** on $\mathbb{R}$ is coarser than the standard topology on $\mathbb{R}$.

5. The finite complement topology is not finer than the lower limit topology on $\mathbb{R}$ since no basis element of the finite complement topology (which are all co-finite sets) can fit inside a lower limit basis element $[a, b)$ (which is uncountably infinite). Conversely, the lower limit topology is finer than the finite complement topology since for any co-finite set and any point within it, we can find a lower limit basis element $[a, b)$ that fits inside the co-finite set. Thus, the lower limit topology is strictly finer than the finite complement topology.