Order, Product & Subpace topologies

Order Relations

An order relation on a set X is a relation $<$ satisfying for all $a, b, c \in X$:

1. (Comparability) Either $a < b$, $a = b$, or $a > b$.
2. (Transitivity) If $a < b$ and $b < c$, then $a < c$.
3. (Non-reflexivity) $a \nless a$.

Examples

1. The usual order on $\mathbb{R}$.
2. The (dictionary order) lexicographic order on $\mathbb{R}^2$: $(a_1, a_2) < (b_1, b_2)$ if either $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$.
3. The dictionary order on words: $a < b$ if either the first letter where they differ is earlier in the alphabet in a than in b, or a is a prefix of b.
4. The parabolic order on $\mathbb{R}^2$ below:

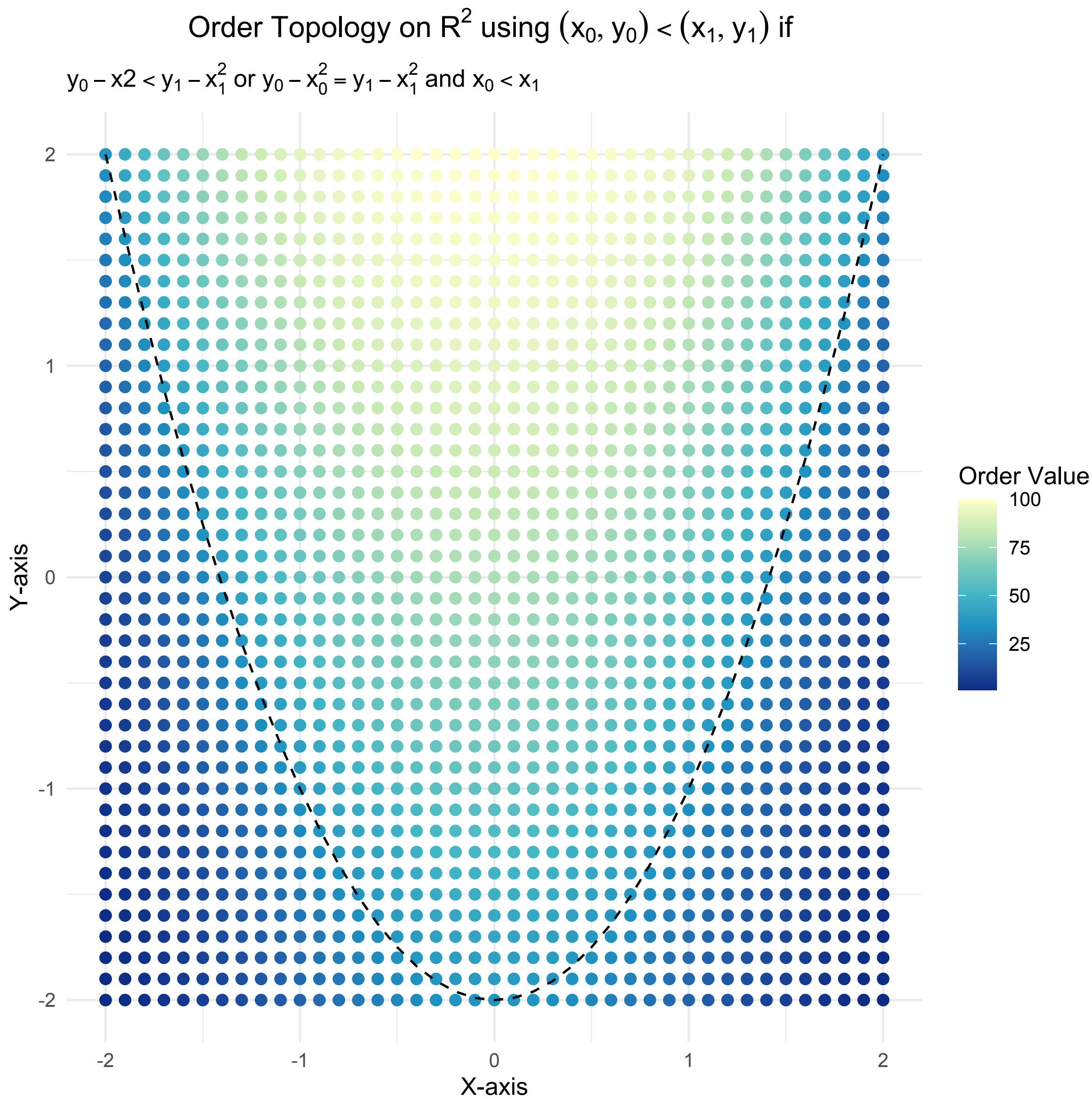

Figure 1: parabolic order

Order Topology

Given a set X with an order relation $<$, the *order topology* on X is generated by the basis consisting of all open intervals $(a, b) = \{x \in X : a < x < b\}$, along with intervals of the form $[a_0, b) = \{x \in X : x < b\}$ if a_0 is the least element of X, and intervals of the form

$(a, b_0] = \{x \in X : a < x\}$ if b_0 is the greatest element of X.

Examples

1. The order topology on $\mathbb{R}$ is the standard topology.
2. The (lexicographic) dictionary order topology on $\mathbb{R}^2$ has two typs of open sets:
    1. Open intervals of the form $(a \times b, c \times d)$ for $a < c$
    2. Vertical open intervals of the form $(a \times b, a \times d)$ for $b < d$

Questions

1. Why do sets of the second type form a basis?
2. What should *rays* be in this topology?

Product Topology

Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, the *product topology* on $X \times Y$ is generated by the basis consisting of all products of open sets $U \times V$ where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$.

Examples

1. The product topology on $\mathbb{R} \times \mathbb{R}$ is the standard topology on $\mathbb{R}^2$.
2. The product topology on $\mathbb{R} \times \mathbb{R}$ where $\mathbb{R}$ has the standard topology and $\mathbb{R}$ has the discrete topology.
3. We will see that the torus inherits a product topology from $S^1 \times S^1$. (First, we need to define the topology on S^1.)
4. The product topology on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the standard topology on $\mathbb{R}^3$.

Question

1. Is the bases $\mathcal{B}$ actually the entire topology? (No, unions of rectangles are not rectangles.)

Main Theorems

Theorem 1 : A (another) basis for the product topology on $X \times Y$ is given by the collection of all products of basis elements $B_X \times B_Y$ where B_X is a basis element for X and B_Y is a basis element for Y.

Proof : Recall, that $\mathcal{B}$ is a basis for a topology on X if for every point $x \in X$ and every open neighborhood U of x, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Let $x \in X$ and assume U is an open neighborhood of x. Then by definition of the product topology, there is a basis element $U_X \times U_Y$ such that $x \in U_X \times U_Y \subseteq U$. Since B_X is a basis for X, there is a basis element B_X such that $x_X \in B_X \subseteq U_X$. Similarly, there is a basis element B_Y such that $x_Y \in B_Y \subseteq U_Y$. Thus, we have that

$$x \in B_X \times B_Y \subseteq U_X \times U_Y \subseteq U$$

as desired. $\square$

Subspace Topology

Given a topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$, the *subspace topology* on Y is all sets of the form $U \cap Y$ where $U \in \mathcal{T}$. In other words, a set $V \subseteq Y$ is open in the subspace topology if there is an open set U in X such that $V = U \cap Y$.

Why is this a topology?

1. $Y = X \cap Y$ is open in the subspace topology since X is open in X.
2. $\phi = \phi \cap Y$ is open in the subspace topology since ϕ is open in X.
3. Let $\{V_\alpha\}$ be a collection of open sets in the subspace topology. Then for each α, there is an open set U_α in X such that $V_\alpha = U_\alpha \cap Y$. Thus,

$$\bigcup_\alpha V_\alpha = \bigcup_\alpha (U_\alpha \cap Y) = \left(\bigcup_\alpha U_\alpha \right) \cap Y$$

   which is open in the subspace topology since $\bigcup_\alpha U_\alpha$ is open in X.
4. Let $V_1, V_2, \dots, V_n$ be a finite collection of open sets in the subspace topology. Then for each i, there is an open set U_i in X such that $V_i = U_i \cap Y$. Thus,

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n (U_i \cap Y) = \left(\bigcap_{i=1}^n U_i \right) \cap Y$$

   which is open in the subspace topology since $\bigcap_{i=1}^n U_i$ is open in X.

It follows that the intersection of basis elements forms a basis for the subspace topology.

Terminology

With subspaces, we need to keep track of two different types of open sets. If $Y \subseteq X$, we say U is open in Y if U is open in the subspace topology on Y. We say U is open in X if U is open in the topology on X.

Examples

1. Let $X = \mathbb{R}$ with the standard topology and let $Y = [0, 1]$. Then the open sets in the subspace topology on Y are all sets of the form $(a, b) \cap [0, 1]$ where (a, b) is an open interval in $\mathbb{R}$. Thus, the open sets in the subspace topology on Y are:

    1. The empty set
    2. Intervals of the form (a, b) where $0 < a < b < 1$
    3. Intervals of the form $[0, b)$ where $0 < b \leq 1$
    4. Intervals of the form $(a, 1]$ where $0 \leq a < 1$
    5. The entire set $[0, 1]$

2. Let $X = \mathbb{R}^2$ with the standard topology and let Y be the x-axis. Then the open sets in the subspace topology on Y are all sets of the form $U \cap Y$ where U is an open set in $\mathbb{R}^2$. Since open sets in $\mathbb{R}^2$ are unions of open balls, it follows that the open sets in the subspace topology on Y are unions of open intervals on the x-axis. Thus, the subspace topology on Y is the standard topology on $\mathbb{R}$.

3. Let $X = \mathbb{R}$ with the standard topology and let $Y = \mathbb{Q}$ be the set of rational numbers. Then the open sets in the subspace topology on Y are all sets of the form $(a, b) \cap \mathbb{Q}$ where (a, b) is an open interval in $\mathbb{R}$. Thus, the open sets in the subspace topology on Y are unions of open intervals (of rationals) with rational endpoints.

4. Let $X = \mathbb{R}$ with the standard topology and let $Y = \mathbb{Z}$ be the set of integers. Then the open sets in the subspace topology on Y are all sets of the form $(a, b) \cap \mathbb{Z}$ where (a, b) is an open interval in $\mathbb{R}$. Thus, the open sets in the subspace topology on Y are all subsets of $\mathbb{Z}$ since for any subset of integers, we can find an open interval in $\mathbb{R}$ that contains exactly those integers. This is the discrete topology on $\mathbb{Z}$.

5. Let $X = \mathbb{R}$ with the standard topology and let $Y = [0, 1) \cup \{2\}$. Then the open sets in the subspace topology on Y are all sets of the form $(a, b) \cap Y$ where (a, b) is an open interval in $\mathbb{R}$. Thus, the open sets in the subspace topology on Y are:

    1. The empty set
    2. Intervals of the form (a, b) where $0 < a < b < 1$
    3. Intervals of the form $[0, b)$ where $0 < b \leq 1$
    4. Intervals of the form $(a, 1]$ where $0 \leq a < 1$
    5. The entire set $[0, 1) \cup \{2\}$

6. The singleton set $\{2\}$
7. Unions of the above sets.

6. Let $X = \mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$ with the subspace topology inherited from the standard topology on $\mathbb{R}^2$. Then the open sets in the subspace topology on X are all sets of the form $U \cap \mathbb{S}^1$ where U is an open set in $\mathbb{R}^2$. Since open sets in $\mathbb{R}^2$ are unions of open balls, it follows that the open sets in the subspace topology on X are unions of open arcs on the circle.

7. Let $X = \mathbb{S}^1 \times \mathbb{S}^1$ be the torus with the product topology inherited from the standard topology on $\mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4$. Then the open sets in the product topology on X are all sets of the form $U_1 \times U_2$ where U_1 and U_2 are open sets in $\mathbb{S}^1$. Since open sets in $\mathbb{S}^1$ are unions of open arcs on the circle, it follows that the open sets in the product topology on X are unions of products of open arcs on the two circles.

8. Let X be the torus with the subspace topology inherited from the standard topology on $\mathbb{R}^3$ via the usual embedding of the torus in $\mathbb{R}^3$. Then the open sets in the subspace topology on X are all sets of the form $U \cap X$ where U is an open set in $\mathbb{R}^3$. Since open sets in $\mathbb{R}^3$ are unions of open balls, it follows that the open sets in the subspace topology on X are unions of open "patches" on the torus. It follows from Theorem 3 below that this topology is the same as the product topology on X.

Some basic Theorems

Lemma 2 : If $U \subseteq Y \subseteq X$, and U is open in Y and Y is open in X, then U is open in X.

Proof : Since U is open in Y, there is an open set V in X such that $U = V \cap Y$. Since Y is open in X, we have that $U = V \cap Y$ is open in X as the intersection of two open sets. $\square$

Theorem 3 : If $A \subset X$ and $B \subset Y$, then the subspace topology on $A \times B$ as a subset of $X \times Y$ is the same as the product topology on $A \times B$ where A and B have the subspace topologies inherited from X and Y, respectively.

Proof : Let $\mathcal{T}_1$ be the subspace topology on $A \times B$ as a subset of $X \times Y$ and let $\mathcal{T}_2$ be the product topology on $A \times B$ where A and B have the subspace topologies inherited from X and Y, respectively. We will show that $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

($\mathcal{T}_1 \subseteq \mathcal{T}_2$) Let $U \in \mathcal{T}_1$. Then there is an open set V in $X \times Y$ such that $U = V \cap (A \times B)$. Since V is open in the product topology on $X \times Y$, there is a basis element $U_X \times U_Y$ such that $x \in U_X \times U_Y \subseteq V$. Thus,

$$U = V \cap (A \times B) \supseteq (U_X \times U_Y) \cap (A \times B) = (U_X \cap A) \times (U_Y \cap B)$$

Since U_X is open in X, we have that $U_X \cap A$ is open in the subspace topology on A. Similarly, $U_Y \cap B$ is open in the subspace topology on B. Thus, $(U_X \cap A) \times (U_Y \cap B)$ is a basis element for the product topology on $A \times B$, and so U is open in $\mathcal{T}_2$.

($\mathcal{T}_2 \subseteq \mathcal{T}_1$) Let $U \in \mathcal{T}_2$. Then there are open sets U_A in A and U_B in B such that $U = U_A \times U_B$. Since U_A is open in the subspace topology on A, there is an open set V_A in X such that $U_A = V_A \cap A$. Similarly, there is an open set V_B in Y such that $U_B = V_B \cap B$. Thus,

$$U = U_A \times U_B = (V_A \cap A) \times (V_B \cap B) = (V_A \times V_B) \cap (A \times B)$$

Since V_A is open in X and V_B is open in Y, we have that $V_A \times V_B$ is open in the product topology on $X \times Y$. Thus, U is open in $\mathcal{T}_1$. $\square$