# Closed Sets & Limit Points

## Closed Sets

A set $C \subseteq X$ is **closed** if its complement $X - C$ is open.

### Examples

1. In the discrete topology, every set is closed.
2. In the indiscrete topology, only $\emptyset$ and $X$ are closed.
3. In the standard topology on $\mathbb{R}$, the set [a,b] is closed because its complement $(-\infty, a) \cup (b, \infty)$ is open.
4. In the standard topology on $\mathbb{R}^2$, the set $\{(x, y) : x^2 + y^2 = 1\}$ (the unit circle) is closed because its complement is open.
5. In the standard topology on $\mathbb{R}$, the set $\mathbb{Z}$ is closed because its complement $\mathbb{R} - \mathbb{Z}$ is open.
6. For $Y = [1, 2] \cup (3, 4)$ is a subspace of $\mathbb{R}$ with the subspace topology. The set $[1, 2]$ is open in $Y$ since it is the intersection of the open set $(-\infty, 2.5)$ in $\mathbb{R}$ with $Y$. Similarly, $(3, 4)$ is open in $Y$. Therefore, the set $[1, 2]$ is closed in $Y$ since its complement $(3, 4)$ is open in $Y$. Also $(3, 4)$ is closed in $Y$ since its complement $[1, 2]$ is open in $Y$.

The properties of a topology can be rephrased in terms of closed sets.

**Theorem 1.** The following properties of closed sets hold in any topological space $X$: 1. The empty set $\emptyset$ and the entire space $X$ are closed. 2. The intersection of any collection of closed sets is closed. 3. The union of any finite number of closed sets is closed.

### Examples

1. The infinite union $\bigcup_{n=1}^{\infty} [1, 2 - \frac{1}{n}] = [1, 2)$ is not closed in the standard topology on $\mathbb{R}^2$ since its complement $(-\infty, 1) \cup [2, \infty)$ is not open. 2 The infinite intersection $\bigcap_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}] = \{0\}$ is closed in the standard topology on $\mathbb{R}^2$ since its complement $(-\infty, 0) \cup (0, \infty)$ is open.

## Terminology

- A set that is both open and closed is called **clopen**. (ajar?)
- **closed in Y** : we say a set $A \subseteq Y$ is closed in $Y$ if $Y - A$ is open in $Y$.

Similar to how we defined subspace topology, we can define closed sets in a subspace.

**Theorem 2.** Let $Y$ be a subspace of a topological space $X$. A set $C \subseteq Y$ is closed in $Y$ if and only if there exists a closed set $D$ in $X$ such that $C = D \cap Y$.

**Proof.** Suppose $C$ is closed in $Y$. Then $Y - C$ is open in $Y$. By the definition of the subspace topology, there exists an open set $U$ in $X$ such that $Y - C = U \cap Y$. Let $D = X - U$. Then $D$ is closed in $X$ and $C = Y - (Y - C) = Y - (U \cap Y) = (X - U) \cap Y = D \cap Y$.

Conversely, suppose there exists a closed set $D$ in $X$ such that $C = D \cap Y$. Let $U = X - D$. Then $U$ is open in $X$ and $Y - C = Y - (D \cap Y) = Y \cap (X - D) = Y \cap U$. Thus, $Y - C$ is open in $Y$, and so $C$ is closed in $Y$. ■

## Terminology

- **interior** : The interior of a set $A$ in a topological space $X$ is the largest open set contained in $A$, denoted by $\text{int}(A)$ or $\text{Int}(A)$ or $\mathring{A}$.
- **closure** : The closure of a set $A$ in a topological space $X$ is the intersection of all closed sets containing $A$, denoted by $\overline{A}$ or $\text{Cl}(A)$.
- **closure** : This is also the smallest closed set containing $A$.

How closures work with subspaces:

**Theorem 3.** Let $Y$ be a subspace of a topological space $X$. For any subset $A$ of $Y$, the closure of $A$ in $Y$ is the intersection of the closure of $A$ in $X$ with $Y$.

**Proof.** Let $x$ be a point in the closure of $A$ in $Y$. Then $x$ is in every closed set in $Y$ that contains $A$. Closed sets in $Y$ are intersections of closed sets in $X$ with $Y$. Thus, $x$ is in every closed set in $X$ that contains $A$, so $x$ is in the closure of $A$ in $X$. Since $x$ is also in $Y$, we have that $x$ is in the intersection of the closure of $A$ in $X$ with $Y$.

Pointwise characterization of closures:

**Theorem 4.** Let $X$ be a topological space, and let $A \subseteq X$. Then $x \in \overline{A}$ if and only if every open set $U$ containing $x$ intersects $A$; that is, $U \cap A \neq \emptyset$.

**Proof.** Suppose $x \in \overline{A}$. Let $U$ be any open set containing $x$. If $U \cap A = \emptyset$, then $A \subseteq X - U$, and since $X - U$ is closed, we have $\overline{A} \subseteq X - U$. This contradicts the fact that $x \in \overline{A}$ and $x \in U$. Thus, $U \cap A \neq \emptyset$. Conversely, suppose that every open set $U$ containing $x$ intersects $A$. If $x \notin \overline{A}$, then there exists a closed set $C$ containing $A$ such that $x \notin C$. Let $U = X - C$.

Then $U$ is an open set containing $x$, but $U \cap A = \emptyset$, contradicting our assumption. Therefore, $x \in \overline{A}$. ■

**Contrapositive Proof.** Let $x \notin \overline{A}$. Then there exists a closed set $C$ containing $A$ such that $x \notin C$. Let $U = X - C$. Then $U$ is an open set containing $x$, but $U \cap A = \emptyset$. Thus, there exists an open set $U$ containing $x$ that does not intersect $A$. Conversely, suppose there exists an open set $U$ containing $x$ such that $U \cap A = \emptyset$. Then $A \subseteq X - U$, and since $X - U$ is closed, we have $\overline{A} \subseteq X - U$. This implies that $x \notin \overline{A}$. ■

### Examples

1. In the standard topology on $\mathbb{R}$, the closure of the open interval (a,b) is the closed interval [a,b].
2. In the standard topology on $\mathbb{R}$, the closure of the set $\mathbb{Q}$ of rational numbers is $\mathbb{R}$ since every open interval in $\mathbb{R}$ contains rational numbers.
3. In the standard topology on $\mathbb{R}^2$, the closure of the set $\{(x, y) : x^2 + y^2 < 1\}$ (the interior of the unit circle) is the set $\{(x, y) : x^2 + y^2 \leq 1\}$ (the unit disk).
4. Let $X = \mathbb{R}$ and let $A = (0, 5]$ Then $\overline{A} = [0, 5]$ in the standard topology on $\mathbb{R}$ since every open set containing 0 intersects $A$.
5. Let $X = \mathbb{R}$ with $B = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Then $\overline{B} = B \cup \{0\}$ in the standard topology on $\mathbb{R}$ since every open set containing 0 intersects $B$.

## Limit Points

A point $x \in X$ is a **limit point** of a set $A \subseteq X$ if every open set $U$ containing $x$ intersects $A$ in some point other than $x$ itself; that is, $(U - \{x\}) \cap A \neq \emptyset$.

### Examples

1. In the standard topology on $\mathbb{R}$, every point in the open interval (a,b) is a limit point of (a,b).
2. In the standard topology on $\mathbb{R}$, the point $a$ is a limit point of the set (a,b) since every open interval containing $a$ intersects (a,b) in some point other than $a$ itself.
3. In the standard topology on $\mathbb{R}$, if $A = (0, 1]$, then 0 is a limit point of $A$.
4. What are the limit points of the set $B$ above in example 5 of closures? Answer: 0 is the only limit point of $B$ since every open set containing 0 intersects $B$ in some point other than 0 itself. No other point in $B$ is a limit point since we can find an open set around any other point that does not intersect $B$ in any other point.

**Theorem 5.** Let $X$ be a topological space, and let $A \subseteq X$. Then $x \in \overline{A}$ if and only if $x$ is a limit point of $A$ or $x \in A$. That is, $A \cup A' = \overline{A}$ where $A'$ is the set of limit points of $A$. iiii
**Proof.** Suppose $x \in \overline{A}$. If $x \in A$, we are done. If $x \notin A$, then for every open set $U$ containing $x$, we have $U \cap A \neq \emptyset$ by Theorem 4. Since $x \notin A$, it follows that $(U - \{x\}) \cap A \neq \emptyset$. Thus, $x$ is a limit point of $A$. Conversely, suppose $x$ is a limit point of $A$ or $x \in A$. If $x \in A$, then clearly $x \in \overline{A}$. If $x$ is a limit point of $A$, then for every open set $U$ containing $x$, we have $(U - \{x\}) \cap A \neq \emptyset$, which implies that $U \cap A \neq \emptyset$. By Theorem 4, this means that $x \in \overline{A}$. ■

**Corollary**. A set $A$ is closed if and only if it contains all its limit points; that is, if $A' \subseteq A$.

**Proof.** Suppose $A$ is closed. Then $\overline{A} = A = A \cup A'$ so $A' \subseteq A$. Conversely, suppose $A' \subseteq A$. Then $\overline{A} = A \cup A' = A$ so $A$ is closed. ■

### Applications

As our knowledge & language grows, we can prove things more easily/efficiently, for example:

1. In the standard topology on $\mathbb{R}$, a single point set $\{x\}$ has no limit points, so it is closed.
2. The same is true for $\mathbb{Z}$ in the standard topology on $\mathbb{R}$ since no point in $\mathbb{R}$ is a limit point of $\mathbb{Z}$. Thus, $\mathbb{Z}$ is closed in $\mathbb{R}$.

## Hausdorff Spaces

General topological spaces can be "weird". For example, not all one-point subsets are closed in every topology.

**Exercise.** Build an example of a topology on a three-point set $X = \{a, b, c\}$ where the one-point set $\{a\}$ is not closed.

Convergence of sequences can also be "weird" in general topological spaces. For example, a sequence can converge to more than one point. First, we need a definition of convergence in topological spaces.

**Definition.** A sequence $(x_n)$ in a topological space $X$ **converges** to a point $x \in X$ if for every open set $U$ containing $x$, there exists an integer $N$ such that for all $n \geq N$, $x_n \in U$.

This definition generalizes the usual definition of convergence in metric spaces. (Here, we avoid using $\epsilon$'s and $\delta$'s.)

**Exercise.** Build an example of a topology on a three-point set $X = \{a, b, c\}$ where a sequence converges to two different points.

To avoid these "weird" situations, we often work with a special* class of topological spaces called **Hausdorff spaces**.

**Definition.** A topological space $X$ is a **Hausdorff space** if for every pair of distinct points $x, y \in X$, there exist open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

One can remember the Hausdorff condition by thinking of $U$ and $V$ as "neighborhoods" that "House-off" the points $x$ and $y$ from each other.

## Examples

1. Any metric space is a Hausdorff space. (Details later.)
2. The discrete topology on any set is a Hausdorff space.
3. The finite complement topology on an infinite set is *not a Hausdorff space*. Why?

**Comment** The Hausdorff condition is also called the $T_2$ axiom. There are other separation axioms ($T_0$, $T_1$, $T_3$, $T_4$, etc.) that impose different levels of "separability" on topological spaces. We will not discuss these much, but the Hausdorff condition strikes a good balance: it is strong enough to avoid many pathological situations, yet weak enough that most topological spaces we encounter in practice are Hausdorff. This balance is often an element of choice in topology; too strong a condition excludes too many spaces, while too weak a condition allows too many pathological cases.

**Theorem 6.** In a Hausdorff space, every one-point set $\{x\}$ is closed.

**Proof.** Apply the Hausdorff condition with $y$ being any point other than $x$. Then there exist open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Since this is true for any point $y \neq x$, we have that $X - \{x\} = \bigcup_{y \neq x} V_y$ where each $V_y$ is an open set containing $y$ and disjoint from some open set containing $x$. Thus, $X - \{x\}$ is open, so $\{x\}$ is closed. ■

The separation axiom $T_1$ is that every finite set is closed. Theorem 6 shows that Hausdorff spaces satisfy the $T_1$ axiom.

**Corollary.** In a Hausdorff space, every finite set is closed.

**Proof.** A finite set is a finite union of one-point sets, and finite unions of closed sets are closed. ■

Finally, we can show that sequences in Hausdorff spaces behave more like sequences in metric spaces.

**Theorem 7.** In a Hausdorff space, a sequence converges to at most one point.

**Proof.** Suppose a sequence $(x_n)$ converges to both $x$ and $y$ where $x \neq y$. By the Hausdorff condition, there exist open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Since $(x_n)$ converges to $x$, there exists an integer $N_1$ such that for all $n \geq N_1$, $x_n \in U$. Similarly, since $(x_n)$ converges to $y$, there exists an integer $N_2$ such that for all $n \geq N_2$, $x_n \in V$. Let $N = \max(N_1, N_2)$. Then for all $n \geq N$, we have $x_n \in U$ and $x_n \in V$, which contradicts the fact that $U \cap V = \emptyset$. Therefore, a sequence can converge to at most one point in a Hausdorff space. ■o

Finally, we observe that subspaces of Hausdorff spaces are also Hausdorff, and products of Hausdorff spaces are also Hausdorff.

**Examples**

1. The torus is a Hausdorff space since it is the product of two circles, and the circle is a Hausdorff space as a subspace of $\mathbb{R}^2$ with the standard topology.
2. The Möbius strip is a Hausdorff space since it is a subspace of $\mathbb{R}^3$ with the standard topology.