# Limit Points, Hausdorff Spaces & Continuous Functions

## Limit Points

A point $x \in X$ is a **limit point** of a set $A \subseteq X$ if every open set $U$ containing $x$ intersects $A$ in some point other than $x$ itself; that is, $(U - \{x\}) \cap A \neq \emptyset$.

### Examples

1. In the standard topology on $\mathbb{R}$, every point in the open interval (a,b) is a limit point of (a,b).
2. In the standard topology on $\mathbb{R}$, the point $a$ is a limit point of the set (a,b) since every open interval containing $a$ intersects (a,b) in some point other than $a$ itself.
3. In the standard topology on $\mathbb{R}$, if $A = (0, 1]$, then 0 is a limit point of $A$.
4. What are the limit points of the set $B$ above in example 5 of closures? Answer: 0 is the only limit point of $B$ since every open set containing 0 intersects $B$ in some point other than 0 itself. No other point in $B$ is a limit point since we can find an open set around any other point that does not intersect $B$ in any other point.

**Theorem 5.** Let $X$ be a topological space, and let $A \subseteq X$. Then $x \in \overline{A}$ if and only if $x$ is a limit point of $A$ or $x \in A$. That is, $A \cup A' = \overline{A}$ where $A'$ is the set of limit points of $A$. iiii
**Proof.** Suppose $x \in \overline{A}$. If $x \in A$, we are done. If $x \notin A$, then for every open set $U$ containing $x$, we have $U \cap A \neq \emptyset$ by Theorem 4. Since $x \notin A$, it follows that $(U - \{x\}) \cap A \neq \emptyset$. Thus, $x$ is a limit point of $A$. Conversely, suppose $x$ is a limit point of $A$ or $x \in A$. If $x \in A$, then clearly $x \in \overline{A}$. If $x$ is a limit point of $A$, then for every open set $U$ containing $x$, we have $(U - \{x\}) \cap A \neq \emptyset$, which implies that $U \cap A \neq \emptyset$. By Theorem 4, this means that $x \in \overline{A}$. ■

**Corollary**. A set $A$ is closed if and only if it contains all its limit points; that is, if $A' \subseteq A$.

**Proof.** Suppose $A$ is closed. Then $\overline{A} = A = A \cup A'$ so $A' \subseteq A$. Conversely, suppose $A' \subseteq A$. Then $\overline{A} = A \cup A' = A$ so $A$ is closed. ■

### Applications

As our knowledge & language grows, we can prove things more easily/efficiently, for example:

1. In the standard topology on $\mathbb{R}$, a single point set $\{x\}$ has no limit points, so it is closed.
2. The same is true for $\mathbb{Z}$ in the standard topology on $\mathbb{R}$ since no point in $\mathbb{R}$ is a limit point of $\mathbb{Z}$. Thus, $\mathbb{Z}$ is closed in $\mathbb{R}$.

## Hausdorff Spaces

General topological spaces can be "weird". For example, not all one-point subsets are closed in every topology.

**Exercise.** Build an example of a topology on a three-point set $X = \{a, b, c\}$ where the one-point set $\{a\}$ is not closed.

Convergence of sequences can also be "weird" in general topological spaces. For example, a sequence can converge to more than one point. First, we need a definition of convergence in topological spaces.

**Definition.** A sequence $(x_n)$ in a topological space $X$ **converges** to a point $x \in X$ if for every open set $U$ containing $x$, there exists an integer $N$ such that for all $n \geq N$, $x_n \in U$.

This definition generalizes the usual definition of convergence in metric spaces. (Here, we avoid using $\epsilon$'s and $\delta$'s.)

**Exercise.** Build an example of a topology on a three-point set $X = \{a, b, c\}$ where a sequence converges to two different points.

To avoid these "weird" situations, we often work with a special* class of topological spaces called **Hausdorff spaces**.

**Definition.** A topological space $X$ is a **Hausdorff space** if for every pair of distinct points $x, y \in X$, there exist open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

One can remember the Hausdorff condition by thinking of $U$ and $V$ as "neighborhoods" that "House-off" the points $x$ and $y$ from each other.

### Examples

1. Any metric space is a Hausdorff space. (Details later.)
2. The discrete topology on any set is a Hausdorff space.
3. The finite complement topology on an infinite set is *not a Hausdorff space.* Why?

**Comment** The Hausdorff condition is also called the $T_2$ axiom. There are other separation axioms ($T_0$, $T_1$, $T_3$, $T_4$, etc.) that impose different levels of "separability" on topological spaces. We will not discuss these much, but the Hausdorff condition strikes a good balance: it is strong enough to avoid many pathological situations, yet weak enough that most topological spaces we encounter in practice are Hausdorff. This balance is often an element of choice in topology; too strong a condition excludes too many spaces, while too weak a condition allows too many pathological cases.

**Theorem 6.** In a Hausdorff space, every one-point set $\{x\}$ is closed.

**Proof.** Apply the Hausdorff condition with $y$ being any point other than $x$. Then there exist open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Since this is true for any point $y \neq x$, we have that $X - \{x\} = \bigcup_{y \neq x} V_y$ where each $V_y$ is an open set containing $y$ and disjoint from some open set containing $x$. Thus, $X - \{x\}$ is open, so $\{x\}$ is closed. ■

The separation axiom $T_1$ is that every finite set is closed. Theorem 6 shows that Hausdorff spaces satisfy the $T_1$ axiom.

**Corollary.** In a Hausdorff space, every finite set is closed.

**Proof.** A finite set is a finite union of one-point sets, and finite unions of closed sets are closed. ■

Finally, we can show that sequences in Hausdorff spaces behave more like sequences in metric spaces.

**Theorem 7.** In a Hausdorff space, a sequence converges to at most one point.

**Proof.** Suppose a sequence $(x_n)$ converges to both $x$ and $y$ where $x \neq y$. By the Hausdorff condition, there exist open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Since $(x_n)$ converges to $x$, there exists an integer $N_1$ such that for all $n \geq N_1$, $x_n \in U$. Similarly, since $(x_n)$ converges to $y$, there exists an integer $N_2$ such that for all $n \geq N_2$, $x_n \in V$. Let $N = \max(N_1, N_2)$. Then for all $n \geq N$, we have $x_n \in U$ and $x_n \in V$, which contradicts the fact that $U \cap V = \emptyset$. Therefore, a sequence can converge to at most one point in a Hausdorff space. ■o

Finally, we observe that subspaces of Hausdorff spaces are also Hausdorff, and products of Hausdorff spaces are also Hausdorff.

**Examples**

1. The torus is a Hausdorff space since it is the product of two circles, and the circle is a Hausdorff space as a subspace of $\mathbb{R}^2$ with the standard topology.
2. The Möbius strip is a Hausdorff space since it is a subspace of $\mathbb{R}^3$ with the standard topology.

## Continuous Functions

We now turn our attention to functions between topological spaces. We want a definition of continuity that generalizes the usual $\epsilon$-$\delta$ definition of continuity in metric spaces.

**Definition.** A function $f : X \to Y$ between topological spaces $X$ and $Y$ is **continuous** if for every open set $V \subseteq Y$, the preimage $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open set in $X$.

We show that this definition generalizes the usual definition of continuity of a real-valued function of a real variable.:

**Theorem 8.** A function $f : \mathbb{R} \to \mathbb{R}$ is continuous in the usual $\epsilon$-$\delta$ sense if and only if it is continuous in the topological sense.

**Proof.** Suppose $f$ is continuous in the usual $\epsilon$-$\delta$ sense. Let $V \subseteq \mathbb{R}$ be an open set. For each point $y \in V$, there exists an $\epsilon_y > 0$ such that the open interval $(y - \epsilon_y, y + \epsilon_y) \subseteq V$. Any $x \in f^{-1}(V)$ determines such a $y$, namely $f(x) \in V$, so the previous sentence guarentes that there exists an $\epsilon_{f(x)} > 0$ such that $(f(x) - \epsilon_{f(x)}, f(x) + \epsilon_{f(x)}) \subseteq V$. By the continuity of $f$ at $x$, there exists a $\delta_x > 0$ such that for all $z \in \mathbb{R}$ with $|z - x| < \delta_x$, we have $|f(z) - f(x)| < \epsilon_{f(x)}$. This implies that $f(z) \in (f(x) - \epsilon_{f(x)}, f(x) + \epsilon_{f(x)}) \subseteq V$. Thus, the open interval $(x - \delta_x, x + \delta_x)$ is contained in $f^{-1}(V)$. Since this is true for every $x \in f^{-1}(V)$, we conclude that $f^{-1}(V)$ is open in $\mathbb{R}$. Therefore, $f$ is continuous in the topological sense.

Conversely, suppose $f$ is continuous in the topological sense. Let $x \in \mathbb{R}$ and let $\epsilon > 0$. Consider the open interval $V = (f(x) - \epsilon, f(x) + \epsilon)$. Since $f$ is continuous, the preimage $f^{-1}(V)$ is an open set in $\mathbb{R}$ containing $x$. Therefore, there exists a $\delta > 0$ such that the open interval $(x - \delta, x + \delta) \subseteq f^{-1}(V)$. This means that for all $z \in (x - \delta, x + \delta)$, we have $f(z) \in V$, which implies that $|f(z) - f(x)| < \epsilon$. Thus, $f$ is continuous in the usual $\epsilon$-$\delta$ sense. ■

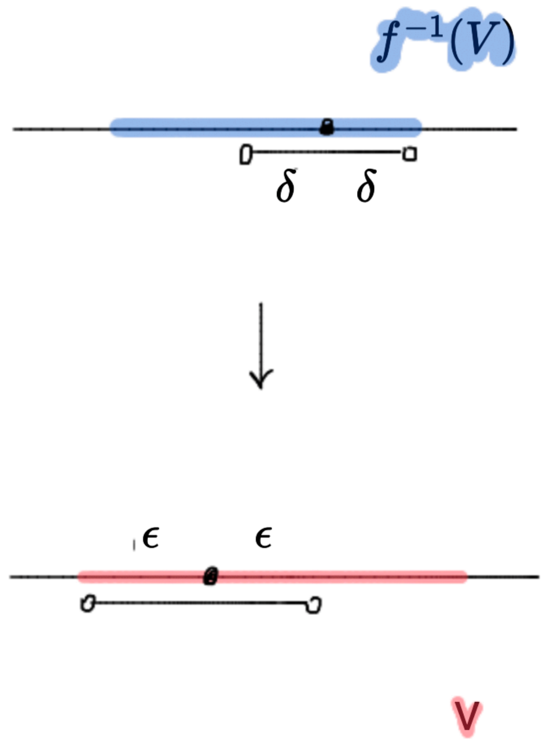


Figure 1: Continuity Diagram

Our definition of continuity generalizes the definitions of continuity for *many* familiar types of functions (curves, maps between surfaces, etc.) since all of these can be viewed as functions between topological spaces.

**Examples**

1. Any function from a set $X$ with the discrete topology to a topological space $Y$ is continuous since every subset of $X$ is open.
2. Any function from a topological space $X$ to a set $Y$ with the trivial topology is continuous since the only open sets in $Y$ are $\emptyset$ and $Y$ itself, and their preimages are $\emptyset$ and $X$, respectively, both of which are open in $X$.
3. The identity function $id_X : X \to X$ is continuous for any topological space $X$ since the preimage of any open set $U \subseteq X$ is $U$ itself, which is open in $X$. 4. The inclusion function $i : A \to X$ defined by $i(a) = a$ for all $a \in A$, where $A$ is a subspace of $X$, is continuous since the preimage of any open set $U \subseteq X$ is $U \cap A$, which is open in the subspace topology on $A$.
4. Constant functions are continuous since the preimage of any open set containing the constant value is the entire domain, which is open, and the preimage of any open set not

containing the constant value is empty, which is also open.

5. *Non-Example.* The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$ is not continuous at $x = 0$ since the preimage of the open set $(-1, 1)$ is $(\infty, 1) \cup \{0\} \cup (1, \infty)$, which is not open in $\mathbb{R}$.
6. *Non-Example.* The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x > 0$ is not continuous at $x = 0$ since the preimage of the open set $(-1, 2)$ is $(-\infty, 0] \cup (0, \infty)$, which is not open in $\mathbb{R}$.
7. The function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2$ is continuous since the preimage of any open set $V \subseteq \mathbb{R}$ is a union of open intervals, which is open in $\mathbb{R}$.
8. Some identity function between different topologies on the same set may not be continuous.

**Theorem 9.**

The following are equivalent for a function $f : X \to Y$ between topological spaces $X$ and $Y$:

1. $f$ is continuous.
2. For every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in $X$.
3. For every subset $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

4. For every $x \in X$ and every neighborhood $V$ of $f(x)$ in $Y$, there exists a neighborhood $U$ of $x$ in $X$ such that $f(U) \subseteq V$.

**Proof.**

(1) $\Rightarrow$ (2): Suppose $f$ is continuous, and let $C \subseteq Y$ be closed. Then $Y - C$ is open in $Y$, so $f^{-1}(Y - C)$ is open in $X$. But $f^{-1}(Y - C) = X - f^{-1}(C)$, so $f^{-1}(C)$ is closed in $X$.

(2) $\Rightarrow$ (1): Suppose that for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in $X$. Let $V \subseteq Y$ be open. Then $Y - V$ is closed in $Y$, so $f^{-1}(Y - V)$ is closed in $X$. But $f^{-1}(Y - V) = X - f^{-1}(V)$, so $f^{-1}(V)$ is open in $X$. Thus, $f$ is continuous.

(3) $\Rightarrow$ (3): Suppose $f$ is continuous, and let $A \subseteq X$. We want to show that $f(\overline{A}) \subseteq \overline{f(A)}$. Let $y \in f(\overline{A})$. Then there exists an $x \in \overline{A}$ such that $f(x) = y$. Since $x \in \overline{A}$, every open set containing $x$ intersects $A$. Let $V$ be any open set containing $y = f(x)$. Since $f$ is continuous, the preimage $f^{-1}(V)$ is an open set containing $x$. Therefore, there exists an element $a \in A$ such that $a \in f^{-1}(V)$, which implies that $f(a) \in V$. Thus, every open set containing $y$ intersects $f(A)$, so $y \in \overline{f(A)}$. Hence, $f(\overline{A}) \subseteq \overline{f(A)}$.

(4) $\Rightarrow$ (1): Suppose that for every subset $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$. Let $V \subseteq Y$ be open, and let $C = Y - V$, which is closed in $Y$. We want to show that $f^{-1}(C)$ is closed in $X$. Let $A = f^{-1}(C)$. Then $f(A) \subseteq C$, so $\overline{f(A)} \subseteq C$. By assumption, we have $f(\overline{A}) \subseteq \overline{f(A)} \subseteq C$. Therefore, $\overline{A} \subseteq f^{-1}(C) = A$. Thus, $A$ is closed in $X$, and hence $f$ is continuous