

Local Neighborhood of a Quiz Solutions

1. State the definitions of

1. a continuous function between topological spaces X and Y .

$f : X \rightarrow Y$ is continuous if for every open set $V \subseteq Y$, the preimage $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open set in X .

2. a closed set in a topological space X .

$A \subseteq X$ is closed if its complement $X \setminus A$ is an open set in X .

3. limit point of a set A in a topological space X .

A point $x \in X$ is a limit point of a set $A \subseteq X$ if every open set U containing x intersects A in some point other than x . In other words, for every open set U containing x , there exists a point $y \in A$ such that $y \neq x$ and $y \in U$.

4. (local formulation of openness) a set U is open if and only if ... (fill in the blank)

A set U is open if and only if for every point $x \in U$, there exists an open set V such that $x \in V \subseteq U$.

5. Hausdorff space.

A topological space X is Hausdorff if for any two distinct points $x, y \in X$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

6. a topological space being *discrete*.

A topological space X is discrete if every subset of X is an open set. Equivalently, the topology on X is the power set of X , meaning that every subset of X is an open set.

7. a basis for a topology on a set X .

A collection \mathcal{B} of subsets of X is a basis for a topology on X if the following conditions hold: 1. For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ such that $x \in B$. 2. If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there is a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. The topology generated by the basis \mathcal{B} consists of all unions of elements of \mathcal{B} .

2. True or False Section

1. True or False: An arbitrary number of intersections of open sets is open. **False.** For example, in \mathbb{R} with the standard topology, the intersection of the open intervals $(-1/n, 1/n)$ for $n = 1, 2, 3, \dots$ is the set $\{0\}$, which is not an open set.

2. True or False: An arbitrary number of intersections of closed sets is closed. **True.** The intersection of any collection of closed sets is closed, because the complement of the intersection is the union of the complements, and the union of open sets is open.
3. True or False: An arbitrary number of unions of closed sets is closed. **False.** For example, in \mathbb{R} with the standard topology, the union of the closed intervals $[n, n+1]$ for $n = 0, 1, 2, \dots$ is the set $[0, \infty)$, which is not a closed set.
4. True or False: An arbitrary number of unions of open sets is open. **True.** The union of any collection of open sets is open by the definition of a topology.
5. True or False: A set A is closed if and only if its complement $X \setminus A$ is open. **True.** By definition, a set A is closed if its complement $X \setminus A$ is open, and conversely, if $X \setminus A$ is open, then A is closed.
6. True or False: A set A is open if and only if its complement $X \setminus A$ is closed. **True.** By definition, a set A is open if its complement $X \setminus A$ is closed, and conversely, if $X \setminus A$ is closed, then A is open.
7. True or False: A set A is closed if and only if it contains all of its limit points. **True.** If A is closed, then its complement is open, and any limit point of A cannot be in the complement, so it must be in A . (Think about this. Do you completely understand the statement?) Conversely, if A contains all of its limit points, then the complement of A cannot contain any limit points of A , which means that the complement of A is open, so A is closed. Again, pause to make sure you understand, as these statements rely on the definitions of limit points and closed sets.
8. True or False: A set A is open if and only if it contains all of its limit points. **False.** For example, in \mathbb{R} with the standard topology, the open interval $(0, 1)$ does not contain the limit point 0 , which is a limit point of the set $(0, 1)$, but $(0, 1)$ is still an open set.

3. Give an example of

1. a continuous function $f : X \rightarrow Y$ between topological spaces X and Y .

The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x$ is continuous with respect to the standard topology on \mathbb{R} , because the preimage of any open set in \mathbb{R} under f is also an open set in \mathbb{R} .

2. a limit point of a set A in a topological space X .

In the topological space \mathbb{R} with the standard topology, the point 0 is a limit point of the set $A = (0, 1)$, because every open interval containing 0 intersects the set $(0, 1)$ in some point other than 0 .

3. a topological space that is Hausdorff.

The real line \mathbb{R} with the standard topology is a Hausdorff space, because for any two distinct points $x, y \in \mathbb{R}$, we can choose disjoint open intervals $U = (x - \epsilon, x + \epsilon)$ and $V = (y - \epsilon, y + \epsilon)$ for some $\epsilon > 0$ small enough such that U and V are disjoint and contain x and y , respectively.

@ Short answer questions 1. Let X be a topological space and let $A \subseteq X$. Prove that \overline{A} is closed.

Recall that \overline{A} is defined as the intersection of all closed sets containing A . Since the intersection of any collection of closed sets is closed, it follows that \overline{A} is closed. 1. Find the closure of the graph of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. (Use Desmos if you like!)

The graph of the function f consists of the points $(x, \sin(1/x))$ for $x \neq 0$ and the point $(0, 0)$. As x approaches 0, $\sin(1/x)$ oscillates between -1 and 1 . Therefore, the closure of the graph of f includes all points of the form $(0, y)$ where $y \in [-1, 1]$, in addition to the points on the graph itself. Thus, the closure of the graph of f is:

$$\overline{\text{Graph}(f)} = \{(x, \sin(1/x)) : x \neq 0\} \cup \{(0, y) : y \in [-1, 1]\}.$$

1. Show the lower limit topology contains a proper subset that is both open and closed.

In the lower limit topology on \mathbb{R} , the basis consists of half-open intervals of the form $[a, b)$ where $a < b$. Consider the set $A = [0, 1)$. This set is open in the lower limit topology because it is a basis element. To show that A is also closed, we need to show that its complement $\mathbb{R} \setminus A = (-\infty, 0) \cup [1, \infty)$ is open. The set $(-\infty, 0)$ can be expressed as a union of basis elements of the form $[a, b)$ with $b < 0$, and the set $[1, \infty)$ can be expressed as a union of basis elements of the form $[a, b)$ with $a \geq 1$. Therefore, both $(-\infty, 0)$ and $[1, \infty)$ are open sets in the lower limit topology, which means that their union $\mathbb{R} \setminus A$ is also open. Hence, A is both open and closed in the lower limit topology. 1. Do continuous functions preserve limit points? That is, if $f : X \rightarrow Y$ is continuous and x is a limit point of a set $A \subseteq X$, is $f(x)$ a limit point of $f(A)$? Prove or give a counterexample.

No. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 0$ for all $x \in \mathbb{R}$. This function is continuous. Now, let $A = (0, 1)$, which has 0 as a limit point. However, $f(A) = \{0\}$, and the point 0 is not a limit point of the set $\{0\}$, since there are no points in $\{0\}$ other than 0 itself. Therefore, this example shows that continuous functions do not necessarily preserve limit points. 1. Let X be a topological space and let $A \subseteq X$. Prove that $\overline{A} = A \cup A'$, where A' is the set of limit points of A .

To prove that $\overline{A} = A \cup A'$, we need to show two inclusions: $\overline{A} \subseteq A \cup A'$ and $A \cup A' \subseteq \overline{A}$. - First, we show that $\overline{A} \subseteq A \cup A'$. Let $x \in \overline{A}$. If $x \in A$, then we are done. If $x \notin A$, then since x is in the closure of A , every open set containing x must intersect A . This means that x is a limit point of A , so $x \in A'$. Therefore, in either case, $x \in A \cup A'$, and we have $\overline{A} \subseteq A \cup A'$. - Next, we show that $A \cup A' \subseteq \overline{A}$. Let $x \in A \cup A'$. If $x \in A$, then since $A \subseteq \overline{A}$, we have $x \in \overline{A}$. If $x \in A'$, then x is a limit point of A , which means that every open set containing x intersects A . This implies that x is in the closure of A , so $x \in \overline{A}$. Therefore, in either case, $x \in \overline{A}$, and we have $A \cup A' \subseteq \overline{A}$. Combining both inclusions, we conclude that $\overline{A} = A \cup A'$. 1. Let X be a topological space and let $A \subseteq X$. Prove that \overline{A} is the smallest closed set containing A . That is, if C is a closed set such that $A \subseteq C$, then $\overline{A} \subseteq C$.

By definition, \overline{A} is the intersection of all closed sets containing A . Let C be any closed set such that $A \subseteq C$. Since \overline{A} is the intersection of all closed sets containing A , it must be the case that $\overline{A} \subseteq C$, because C is one of the closed sets in the intersection. Therefore, \overline{A} is the smallest closed set containing A . 1. Is \mathbb{Q} a closed set in \mathbb{R} ? Prove your answer.

No, \mathbb{Q} is not a closed set in \mathbb{R} . To see this, we need to check if the complement of \mathbb{Q} , which is the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$, is an open set. The set of irrational numbers is not open in the standard topology on \mathbb{R} because it does not contain any open interval. For any irrational number x , any open interval around x will contain both rational and irrational numbers, so there is no open interval that is entirely contained within the irrationals. Therefore, the complement of \mathbb{Q} is not open, which means that \mathbb{Q} is not closed in \mathbb{R} .