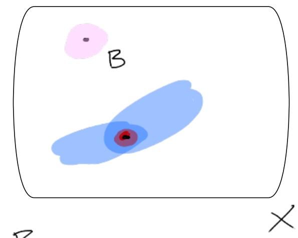
Basis For a Topology

Too many open sets to describe a topology — we use a "basis"

A basis a collection of open sets in which every open set is a union of basis elts

Def'n For a set X, a collection of subsets of X $\mathcal{B}$ is a basis for a topology on X if:



① (Coverage) $\forall x \in X$ $\exists$ at least one basis elt $B \in \mathcal{B}$ s.t. $x \in B$

② (nesting property) If $x \in B_1 \cap B_2 \Rightarrow \exists B_3 \subset B_1 \cap B_2$ s.t. $x \in B_3$

munkres: $\subseteq$ = $\subset$

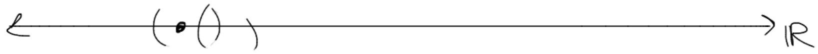
A basis defines a topology by declaring:

★ $U \subset X$ open iff $\forall x \in U$ $\exists B_x \subseteq U$ s.t. $x \in B_x \in \mathcal{B}$

Note: Basis elts are open (replace U w/ a basis elt B_x)

examples of bases

① Standard topology on $\mathbb{R}$: the collection of all open intervals (a,b) where $a<b$.



here, intersection of two basis elements IS another basis element

② Discrete Topology on X (Every set is open). Basis = {set of all singletons} = $\{x\}$ s.t. $x \in X$

$[1,7)$ = union of all single $\{x\}$, $x \in [1,7)$

③ Lower limit topology on $\mathbb{R}$: set of all half-open intervals $[a,b)$ w/ $a<b$.

$[1,5) \cap [0,3) = [1,3)$

④ $B^{\circ} = \{$all open balls (disc) in $\mathbb{R}^2\}$.

↓ use Eucl. dist in define balls

① cover! $\forall x \in \mathbb{R}^2$, $\exists$ open ball containing it

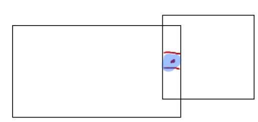
② nesting property:

⑤ $B^{\square} = \{$all open squares in $\mathbb{R}^2\}$.

① $\forall x \in \mathbb{R}^2$ $\exists$ open square containing it

② nesting property.

any two squares intersect in a rectangle
and any rectangle contains a square



A Basis Really Does Form a topology

① $\phi \in \mathcal{T}$ (ϕ contains nothing so is (vacuously) a basis elt)
$X \in \mathcal{T}$ (X is open, b/c $\underbrace{\forall x \in X, \exists B \ni x \text{ \& } B \subseteq X}_{\Rightarrow X \text{ is open}}$)

contains

② why are arbitrary unions of open sets open.
Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets (in the top defined by basis $\mathcal{B}$)

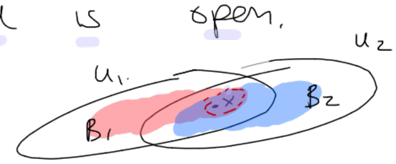
show $U = \bigcup_{\alpha \in A} U_\alpha$ is open.

Let $x \in U$. show $\exists$ basis elt $B_x \ni x$ & $B_x \subseteq U$

we know $x \in U_0$ where $U_0 \in \underbrace{\{U_\alpha\}_{\alpha \in A}}_{\text{open}}$, $U_0 \subseteq U$. $\Rightarrow$ U is open

③ Show Finite intersections of open are open.
Induction: step 1' Let U_1, U_2 be open sets (in the top generated by $\mathcal{B}$)

$U = U_1 \cap U_2$ show U is open.



let $x \in U$ it is in U_1, which is open,
thus $\exists$ basis elt. $B_1 \subset U_1$ & $x \in B_1$

& $x \in U_2 \Rightarrow \exists B_2$ s.t. $x \in B_2 \subseteq U_2$

$\Rightarrow U$ is open

Now $x \in B_1 \cap B_2 \overset{\text{(basis)}}{\Rightarrow} \exists B_3$ s.t. $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U$
induction step is similar

Union lemma: A topology is the union of its basis elts.

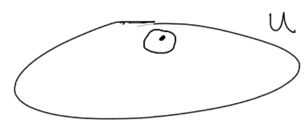
an open set is just a union of basis elements

proof: Let τ = topology on X.

Show $\tau \subseteq \bigcup_{B \in \mathcal{B}} B$ & $\bigcup_{B \subset \mathcal{B}} B \subseteq \tau$

① Let $U \in \tau$. For any! $x \in U$, $\exists$ basis elt $B_x \in \mathcal{B}$ s.t. $x \in B_x \subseteq U$.

thus $U = \bigcup_{x \in U} B_x$ $\left(\Rightarrow \tau \in \bigcup_{B \subset \mathcal{B}} B \right)$ basis elt



② Now let $U \in \bigcup_{B \in \mathcal{B}} B$. Since each B is open, & arbitrary unions of open sets are open

U is open.

$\left(\Rightarrow \bigcup_{B \in B} B \in \tau \right)$

ex $\mathbb{R}^2$

we have two different bases for the std. top: $\underbrace{\text{open circles}}_{\mathcal{B}^{\circ}}$ & $\underbrace{\text{open squares}}_{\mathcal{B}^{\square}}$

By union lemma, std top = union of open circles
std top = union of open squares

unions of open circles give the same topology as open squares`````