Week 3

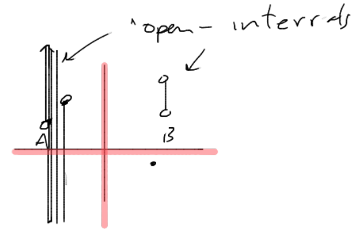
Order, Subspace Topologies

Order Relation. on a set X is a relation $<$ s/t

① Comparability: either $a=b$, $a<b$ or $b<a$

② Transitivity. $a<b$ & $b<c \Rightarrow a<c$

③ Non-reflexivity: $a \not< a$



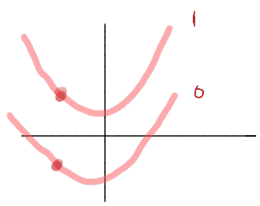
Ex ① usual order on $\mathbb{R}$

② lexicographic order on $\mathbb{R}^2$. $(a_1, a_2) < (b_1, b_2)$ if

$a_1 < b_1$ or $a_1 = b_1$ & $a_2 < b_2$

③ lexicographic ordering on English words: {although in group theory, a word ≡ element in a group}

④ Parabolic ordering on $\mathbb{R}^2$: $(x_0, y_0) < (x_1, y_1)$ if



$\underbrace{y_0 - x_0^2 < y_1 - x_1^2}$ or $\underbrace{y_0 - x_0^2 = y_1 - x_1^2}$ & $x_0 < x_1$

Order Topology:

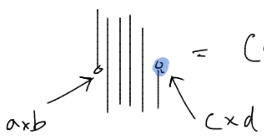
Given a set X w/ order relation $<$ the order top on X is generated by the basis: all open intervals $(a,b) = \{x \in X : a < x < b\}$

along w/ $[a_0, b)$ w/ $a_0 =$ least elt.

$(a, b_0]$ w/ $b_0 =$ greatest elt.

ex ① the order top on $\mathbb{R}$ is the same as the standard top

② lexicographic order on $\mathbb{R}^2$. has two types of open sets

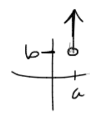


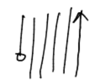
$= (a \times b, c \times d)$ w/ $a < c$

vertical open ints $(a \times b, a \times d)$ $b<d$

questions ① which of these form a basis? vertical ones (Union Lemma)

② How to define rays?, $(a \times b, a \times \infty)$

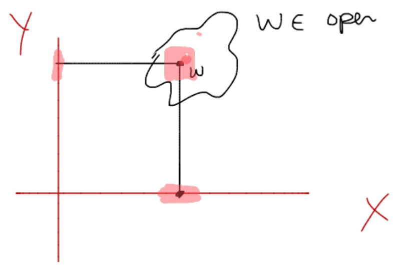




Product Topology

given top. spaces (X, τ_X) & (Y, τ_Y) the product topology on $X \times Y$ is generated by the basis:

all open sets $U \times V$ w/ $U \in \tau_X$, $V \in \tau_Y$

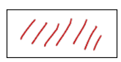


$W \in$ open in Product Top

Question: Is this basis also a topology? (def'n)

– No? union of rectangles not always a rectangle

ex Product top on $\mathbb{R} \times \mathbb{R}$ ($\times \mathbb{R}$) is the same as the std. top on $\mathbb{R}^2$ (^{3}). (

why? basis elts ↑:  basis elt. ↑ : 

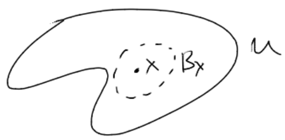
ex we'll see Torus  interits a product topology from $S^1 \times S^1$

unit circle

Main Thm

Another basis for the product top. on $X \times Y$ is: all products of basis $B_X \times B_Y$ w/ B_X, B_Y are bases of X, Y.

proof: Recall: B is a basis for a top on X if



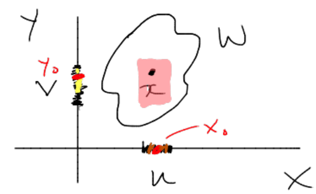
$\forall x \in X$ & and $\forall$ neighborhood U, $\exists\ B_x \in B$ st $x \in B_x \subset U$.

So let $x \in X \times Y$ & let W be a nbhd,

By def of product top $\exists$ a basis elt $U \times V \subset U$ s.t. $x \in U \times V$.

$x = x_0 \times y_0$ & $x_0 \in U$, $y_0 \in V$.

So $x \in B_x \times B_y \subseteq U \times V \subset W$



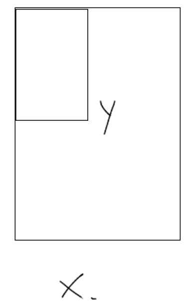
Since B_X is a basis, $\exists\ B_0 \in B_X$ $x \in B_0 \subset U$
Similarly for B_Y

Subspace Topology: (how subsets become top. spaces themselves)

given (X, τ) and a subset $Y \subset X$ the subspace top on Y is all sets:

$U \cap Y$ where $\overbrace{U \in \tau}^{\text{i.e. } U \text{ is open}}$. (i.e., a set $V \subset Y$ is open in subspace top if $\exists$ open set $U \subset X$ s.t. $V = U \cap Y$.)

why is this a topology?



① whole space: Y is open in subspace top b/c: $Y = X \cap Y$

② null space: ϕ is open in sub space top b/c: $\phi = \phi \cap Y$

③ ∞ unions: Let $\{V_\alpha\}$ be a collection of open sets in subspace top

For each α $\exists$ some $V_\alpha = U_\alpha \cap Y$

open in X

our goal was to show the union of all the V-alpha was open. we found a larger open set whose intersection with Y was exactly V-alpha, thus V-alpha is open in SS-top.

$$\bigcup_\alpha V_\alpha = \bigcup_\alpha (U_\alpha \cap Y) = \underbrace{\bigcup_\alpha U_\alpha}_{\text{open by property of top. of } X} \cap Y$$

④ finite intersections: follows similarly (exercise)

Termnology:

If $Y \subset X$ we say

<u>U is open in Y</u>: U is open in subspace top on Y

<u>U is open in X</u>: U is open in top on X.

(ex) $X = \mathbb{R}$ (std. top)

$Y = [0,1]$.

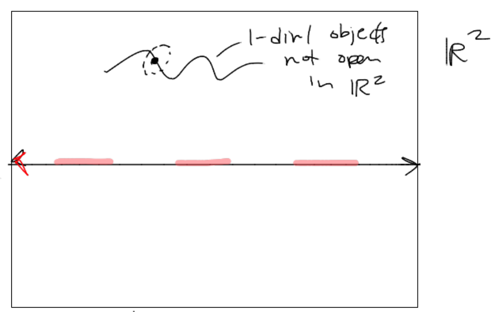
what are open sets in subspace top on Y?

1. $(4,14) \cap [0,1] = \phi$ is thus open!
   (open!)
2. $(-\frac{1}{2},\frac{1}{2}) \cap [0,1] = [0,\frac{1}{2})$ open!
3. $(-\infty,\infty) \cap [0,1] = [0,1]$ open
4. $(\frac{1}{2},\frac{3}{2}) \cap [0,1] = (\frac{1}{2},1]$ open!
5. $(0,1) \cap [0,1] = (0,1)$ open!

(in Subspace Top)

these are all the main types

(2) $X = \mathbb{R}^2$ (std. top)



$Y = x$-axis

Describe the subspace top on Y.

- Y is open: $Y = Y \cap \mathbb{R}^2$
- ϕ is open: $\phi = Y \cap \phi$
- $(-1,1)$ is open: $(-1,1) = Y \cap D'$ open unit disc

unions of these

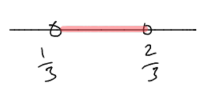
(3) $X = \mathbb{R}$ (std. top)

$Y = \mathbb{Q}$

Describe open sets in subspace top on $\mathbb{Q}$.

$U \subset \mathbb{Q}$ is open $\Leftrightarrow$ $U =$ open set in $\mathbb{R}$ $\cap\ \mathbb{Q}$

$(\frac{1}{3},\frac{2}{3}) \subset \mathbb{Q}$



$(\frac{1}{3},\frac{2}{3}) = (\frac{1}{3},\frac{2}{3}) \cap \mathbb{Q}$

all rationals b/w $\frac{1}{3}$ & $\frac{2}{3}$.

is $\{\frac{1}{2}\}$ open? NO

- any open set in $\mathbb{R}^2$ containing $\frac{1}{2}$ also contains other rationals.