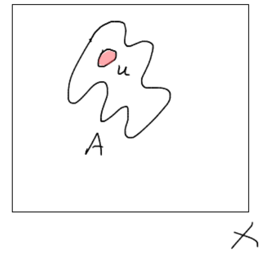
Week 3 Wed

Closed sets & Limit Points

(First: recall subspace top: If $A \subset X$, $U \subset A$ is open if $\underline{U = V \cap A}$ w/ V open in X.



ex) $X = \mathbb{R}$, $A = [0,5) \cup \{12\} \cup (15,20]$

describe some open sets in A.

(i) A is open in A b/c: $A = A \cap \underset{\text{open in } \mathbb{R}}{(-1,21)}$

(ii) ϕ is open in A b/c $\phi = A \cap \phi$

(iii) $(1,2)$ open in A b/c $(1,2) = A \cap (1,2)$

(iv) $(15,17)$ open in A b/c $(15,17) = A \cap (15,17)$

(v) $(15,16)$ open in A $(15,16) = A \cap (13,16)$

(vi) $\boxed{\{12\}}$ open in A $\{12\} = A \cap (11.9, 12.1)$

(vii) $(19,20]$ open in A $(19,20] = A \cap (19, 20.1)$

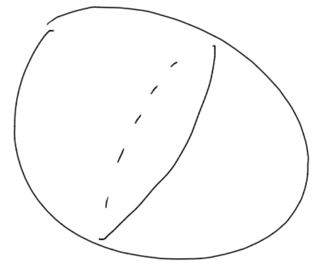
Q: Define Σ_2 as a topological space. Give it the subspace top since it embed in $\mathbb{R}^3$.

$\Sigma_2 =$  2-D surface

① open sets (basis elts)

②

1 open 3-ball



Closed Sets & Limit Points

Def'n A set $C \subseteq X$ is closed if its complement $X - C$ is open.

ex ① In the discrete topology: (where every set is open)

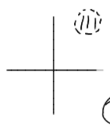
Let C be a subset in X, the $X - C$ is also a set, thus $X-C$ is open

$\Rightarrow C$ is closed. $\Rightarrow$ Every set in discrete is closed.

② In the std. top on $\mathbb{R}$ $[a,b]$ closed intervals are closed sets b/c

$$\mathbb{R} - [a,b] = \underbrace{(-\infty, a) \cup (b, \infty)}_{\text{union is also open}}$$

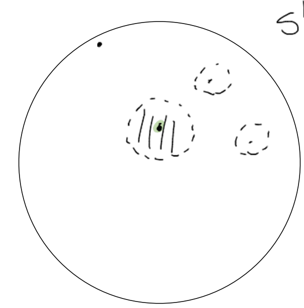
open interval (open in std. top)



③ $(\mathbb{R}^2, T_{std})$. $S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$ is closed.

Must show $\mathbb{R}^2 - S^1$ is open. i.e.,

$\forall x \in \mathbb{R}^2 - S^1$ we need to find an open nbhd containing x but $\underbrace{\text{fully inside } \mathbb{R}^2 - S^1}_{\text{disjoint from } S^1}$



ALT: $\mathbb{R}^2 - S^1 = \{(x,y) \mid x^2 + y^2 > 1\} \cup \{(x,y) \mid x^2 + y^2 < 1\}$

show these are open

④ In $\mathbb{R}$, $\mathbb{Z}$ is closed b/c: The complement of the integers is all open intervals whose endpoints are (n,n+1) for all n in Z. Open intervals are open sets, unions of open sets are still open. Thus the entire complement is open.

Y is neither open nor closed.

⑤ $(\mathbb{R}, T_{std})$, $Y = [1,2] \cup (3,4)$. Is Y closed? (NO)

$[1,2]$ is not open

$(3,4)$ is open

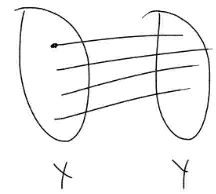
$\mathbb{R} - Y = (-\infty, 1) \cup (2, 3] \cup [4, \infty)$

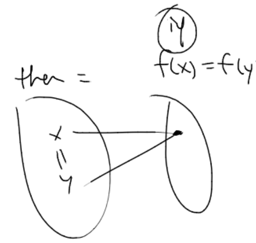
Is this open? NO

Any open nbhd of 3 is not contained in $\mathbb{R} - Y$.

1-1 : injective : $f : X \longrightarrow Y$. is injective if the following is true

$$\boxed{\text{if } f(x) = f(y) \text{ then } x = y}$$





onto : surjectivity $f : X \to Y$ is surjective iff

$$\boxed{\forall\ y \in Y\ \exists\ x \in X \text{ s.t. } f(x) = y.}$$

the properties of any topology can phrased in terms of closed sets ———

usual open properties $\longrightarrow$ closed properties

ϕ is open $\xrightarrow{\text{it's complement is closed}}$ X is closed

X is open $\longrightarrow$ $X - X = \phi$ is closed

arbitrary unions of open are open $\longrightarrow$ Arbitrary intersections of closed are closed

$\bigcup_{\alpha \in A} \overset{\text{open}}{U_\alpha}$ is open

so $\overset{\text{DeMorgan}}{=} \bigcap_{\alpha \in A} \overbrace{X - U_\alpha}^{\text{closed}}$

$X - \bigcup_{\alpha \in A} U_\alpha$ is closed

finite intersection of open sets are open

$\underbrace{\bigcap_{n=1}^{N} \overset{\text{open}}{U_n}}_{\text{open}} \Rightarrow X - \bigcap_{n=1}^{N} U_n$ is closed $\overset{\text{DeMorgan}}{=} \bigcup_{n=1}^{N} \overbrace{X - U_n}^{\text{closed}}$

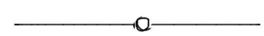
Finite union of closed sets are closed

ex $\bigcup_{n=1}^{\infty} \left[1, 2 - \frac{1}{n}\right] = [1,1] \cup \left[1, 2 - \frac{1}{2}\right] \cup \left[1, 2 - \frac{1}{3}\right] \cup \dots = [1,2)$

not closed

here is an infinite union of closed sets that is not closed

ex $\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right] = \{0\}$



here we took an infinite intersection of closed and we produced a closed set (we expected this from property above. Note {0} is closed not only by this property, but also because its complement is open