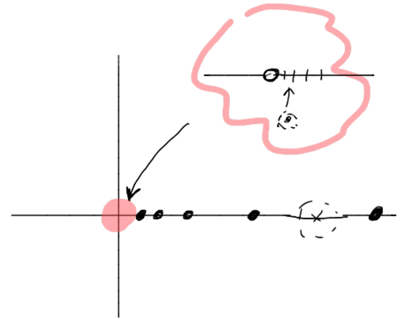
Closure, Limit Points, Hausdorff

warm-up: $S = \{ (\frac{1}{n}, 0) \mid n \in \mathbb{N} \} \subseteq \mathbb{R}^2$



- Is S closed in $\mathbb{R}$? (Is $\mathbb{R}^2 - S$ open?)
- Is $\mathbb{R}^2 - S$ closed in $\mathbb{R}^2$? (No! b/c S isn't open, S is not union of ϵ-balls.)
- Is $S \cup \{(0,0)\}$ closed? Yes, $\mathbb{R}^2 - (S \cup \{0,0\})$ is open

Fact
- $(0,0) \notin S$, so $(0,0) \in \mathbb{R}^2 - S$. If $\mathbb{R}^2 - S$ is open we could find a nbhd of $(0,0)$ contained in $\mathbb{R}^2 - S$. we can't. Any nbhd of $(0,0)$ contains infinitely points of S.

we'll see $\{(0,0)\}$ is a "Limit Point" of S.

Terminology:

- A set can be both open & closed. (Clopen) (ajar) :)

closed in Y ($C \subseteq Y \subseteq X$) means $Y - C$ is open in Y.

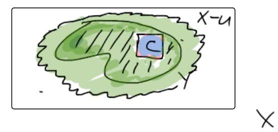
Similar to how we define subspace top we define closed sets in subspace

Thm: $Y \subseteq X$, $C \subseteq Y$ is closed iff $\exists$ closed set D in X s.t.

$$C = D \cap Y$$

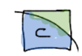
proof: ($\Rightarrow$) Suppose C is closed in Y, thus $Y - C$ is open in Y.

So $Y - C = U \cap Y$ w/ U is open in X. Let $D = X - U$, which must be closed in X.



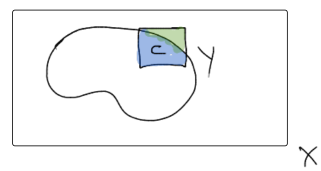
$$C = Y - (Y - C)$$

$$= Y - (U \cap Y) = X - U \cap Y = D \cap Y$$



($\Leftarrow$) Suppose $\exists$ closed D in X s.t. $C = D \cap Y$

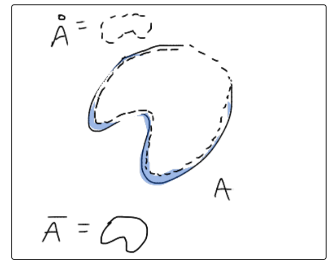
Let $U = X - D$, which is open
(white subset)



$$Y - C = Y - D \cap Y = \underbrace{Y \cap \underbrace{(X - D)}_{\text{open in } X}}_{\text{open in } Y}$$

$\Rightarrow C$ is closed in Y

- interior: interior of A is largest open set contained inside A, int(A), $\mathring{A}$
- closure: closure of A is the intersect of all closed sets containing A. $\bar{A}$, CL(A)
  (smallest closed set containing A)



thm: (How closures work w/ subspaces)
For $Y \subseteq X$, $\forall$ subset $A \subseteq Y$ the closure of A in Y is the intersection of the closure of A in X w/ Y.

X

$A = [\overset{1}{a}, \overset{5}{b}) \subset \mathbb{R}$

$\mathring{A} = (a, b)$

$\bar{A} = [1, 5]$

$[0, 10]$ closed containing $[1, 5)$

ex $X = \mathbb{R}$

$Y = [1, 10) \cup \{15\}$

$A = (9, 10)$

closure of A in Y : intersection of all $\underbrace{\text{closed sets in Y}}_{\text{intersection of closed sets in X w/ Y}}$ that contain A. $[9, 10)$

should be (thm)

$\bar{A} \cap Y$

$= [9, 10] \cap Y = [9, 10)$

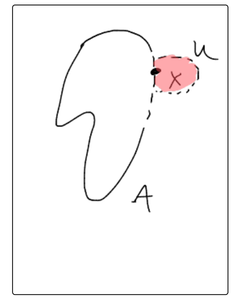
pointwise characterization of closures

thm For $A \subseteq X$

$$x \in \bar{A} \iff \text{every open set } (U) \text{ containing } x \text{ intersects } A, \ U \cap A \neq \phi.$$

all nbhds of x intersect A

proof: ($\Rightarrow$)
let $x \in \bar{A}$, and let U be any nbhd of x



If $U \cap A = \phi$, so $A \subseteq X - U$ (a closed set that contains A)

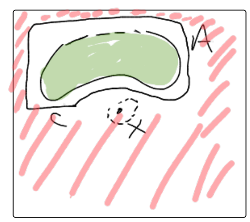
so $\bar{A} \subseteq X - U$. So this contradicts $x \in \bar{A}$

$\Rightarrow U \cap A \neq \phi$

($\Leftarrow$) now suppose every open set U containing x intersects A,

X

If $x \notin \bar{A}$ so $\exists$ some closed set (C) containing A, that doesn't contain x.



$U = X - C$ is open & $x \in X - C$ but ... this contradicts the main assumption since $A \subseteq C$. (i.e., $X - C$ is an open set containing x disjoint from A)

(ex)

- In $(\mathbb{R}, \text{std})$ the closure of any open int. is a closed int.
- In $(\mathbb{R}, \text{std})$, $\overline{\mathbb{Q}} = \mathbb{R}$

R is closed and contains Q, so its immediately a candidate for being Q-closure. Is there a smaller closed subset that contains Q?

the integers,

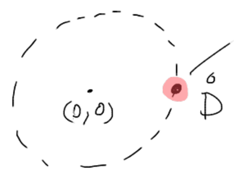
[1,10]

Any reasonable candidate would have to be missing an interval. But any interval contains some rational.

- $(\mathbb{R}^2, \text{std})$. $A = \mathring{D} = \text{unit } \overset{\text{open}}{\text{disk}} = \{(x,y) \mid x^2+y^2<1\}$,

$\overline{\mathring{D}} = \{(x,y) \mid x^2+y^2 \leq 1\}$

(see prev. thm.



$(0,0) \in \overline{\mathring{D}}$ since every nbhd contains ε-disc centered @ (0,0).

$\{(1,0)\} \in \overline{\mathring{D}}$ b/c

$\underbrace{\text{any nbhd}}_{\text{contains basis } \epsilon\text{-disc centered @ } (1,0)}$ of $(1,0)$ intersects $\mathring{D}$ (

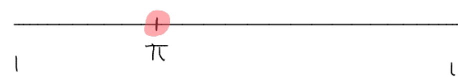
Limit Points

A point $x \in X$ is a limit point of the set $A \subseteq X$ if every nbhd of x intersects A in a point other than x.

(this means that x is the limit of some sequence of points within A)

ex $(\mathbb{R}, \text{std})$ every point in an open interval (a,b) is a limit point of (a,b)

(e.g.) $A = (1,10)$, let $x = \pi$.



ex "the endpoints" of (a,b) are limit points of (a,b)

ex $A = (0,1) \cup \{5\}$.

Is 5 a limit point of A?

ex $\{(0,0)\}$ is a limit point of $S = \{(0, \frac{1}{n}) \mid n \in \mathbb{Z}^+\}$.

we'll see

limit point

$$\overline{A} = A \cup A'$$