Limit Points, Hausdorff Spaces & Continuous Fcns.

Recall:
x is a limit pt of a set A if every nbhd of x intersects A, in a point other than x.

ex $(\mathbb{R}, \text{std})$ $A = [5,13) \cup \{20\}$

(*) $\underline{5}$ is a limit pt of A b/c every nbhd of 5 contains $(5-\epsilon, 5+\epsilon)$ for some $\epsilon > 0$, this interval intersects A in $[5, 5+\epsilon)$.

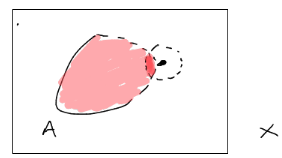
$\underline{20}$ isn't limit pt of A b/c the nbhd of 20 $(19,21) \cap A = \{20\}$

$\underline{13}$ is (replace 5 (*) above w/ 13) $\rightsquigarrow$ $(13-\epsilon, 13) \subset$ Every nbhd of $\{13\} \cap A$

Thm: $A \cup A' = \overline{A}$ (A' = all limit pts of A), $\overline{A}$ = closure of A.

ex $A = [5,13) \cup \{20\}$ ($A' \neq \{13\}$
$\overline{A} = [5,13] \cup \{20\} = A \cup \{13\}$ $A' = [5,13]$

proof: Let $x \in \overline{A}$. If $x \in A$ we're done, so assume $x \notin A$. So $\forall$ nbhd U_x of x, $U_x \cap A \neq \phi$. Also $(U_x - \{x\}) \cap A \neq \phi$. So x is a limit point.



Conversely, let $x \in A \cup A'$. If $x \in A$ (Always $A \subset \overline{A}$) then $x \in \overline{A}$, alternatively if $x \in A'$.

Thus every nbhd of x intersects A in some other point. This implies every nbhd of x intersects A. This last part is equivalent to x being in the closure (Thm).

Cor $A \subseteq X$ is closed $\Leftrightarrow$ A contains its limit pts

Applications: (More language + knowledge) $\Rightarrow$ more efficient (shorter) proofs

① $(\mathbb{R}, \text{std})$ $\{x\}$ is closed b/c it has no limit points.

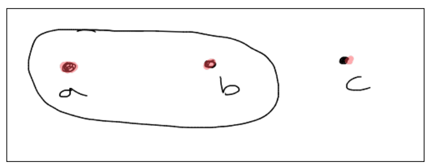
② $\mathbb{Z} \subseteq \mathbb{R}$ $\mathbb{Z}$ is closed b/c $\mathbb{Z}$ contains no limit pts.

Hausdorff Spaces

General topological spaces can be "weird". (Ex, not all $\{x\}$ are closed in every topology)

ex Build a topology on $X = \{a, b, c\}$ where $\{a\}$ is not closed.

(open set $\equiv$ box/circle containing x of x)



$= X$

not a "Hausdorff Top."

This is a top. on $\{a, b, c\}$.

$\{c\}$ is closed

$\{a\}$ is not closed b/c

$X - \{a\} = \{b, c\}$

(no box around $\{b, c\}$.

Def. X is Hausdorff if every pair of points have disjoint nbhds.

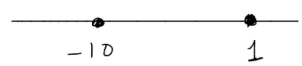
ex $\mathbb{R}$, $\mathbb{R}^n$ w/std topologies are Hausdorff

ex any metric space is Hausdorff

ex F.C.T. on $\mathbb{R}$ is not a Hausdorff Space

Finite Complement Top

$W = \mathbb{R} - \bigcup_{n=1}^{N} \left(\frac{1}{n}\right)$ is a nbhd of $\{-10\}$ & $W \cap V \neq \phi$

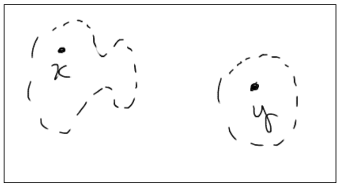


$U = (-\infty, 1) \cup (1, \infty)$ is open yet doesn't contain $\{1\}$

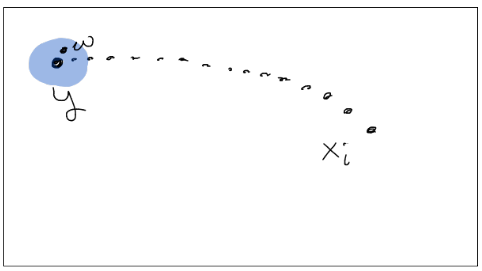
V is a nbhd (in F.C.T.) of $\{1\}$ if $V = (-\infty, 0) \cup (0, \infty)$

thm: In a Hausdorff Space (T_2) Every one point set is closed.* (T_1)

proof: (show $X - \{x\}$ is open, $\forall x \in X$.) Let $x \in X$. Pick $y \neq x \in X$, apply Hausdorff condition $\Rightarrow$ nbhd U_y of y that is disjoint from some nbhd U_x of x. We can do this for all $y \in X - \{x\}$ by Hausdorff cond. This implies $X - \{x\}$ is open.



thm: In a Hausdorff space, a sequence converges to at most one point.



(ex) The torus is a Hausdorff Space (is a product of two circles the circle $S^1 \subset \mathbb{R}^2$ is Hausdorf (its a subspace of a Hausdorff space) & the product of Hausdorff Spaces is Hausdorff)

Continuous Functions

Def'n: $f: X \to Y$ a mapping b/w topologized spaces is cts if.

the preimage of every open set is open,

i.e., $\forall$ open sets $V \subseteq Y$ the preimage

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\} \text{ is open in } X.$$

$\epsilon - \delta$ def'n

> $f: \mathbb{R} \to \mathbb{R}$ is cts, if @ x_0
>
> $\forall \epsilon > 0 \ \exists \delta > 0$ st.
>
> whenever $|x - x_0| < \delta$
>
> then $|f(x) - f(x_0)| < \epsilon$

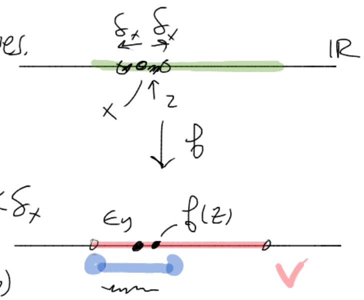
ex $f: \mathbb{R} \to \mathbb{R}$ (set $\to$ set), $f(x) = x$. $f: (\mathbb{R}, std) \longrightarrow (\mathbb{R}, std)$ is cts

w/ $f: (\mathbb{R}, std) \longrightarrow (\mathbb{R}, \text{Lower Limit})$ $[a,b)$ is not cts

thm: A function $f: \mathbb{R} \to \mathbb{R}$ is cts in the usual $\epsilon - \delta$ sense $\iff$ its cts in the topologized sense

$(\Rightarrow)$ Suppose f is ϵ-δ-cts. Let $V \subset \mathbb{R}$ be an open set. (show $f^{-1}(V)$ is open)
For every $y \in V$ $\exists\ \epsilon_y > 0$ st. $(y - \epsilon_y, y + \epsilon_y) \subseteq V$. Every $x \in f^{-1}(V)$ has a $y = f(x) \in V$ so the previous ↑ applies.

$(f(x) - \epsilon_{f(x)}, f(x) + \epsilon_{f(x)})$ is open nbhd in V.



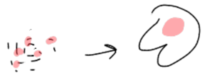
By continuity of f, $\exists\ \delta_x > 0$ st. $\forall z \in \mathbb{R}$ st. $|z - x| < \delta_x$
$|f(z) - f(x)| < \epsilon_{f(x)}$ $\Rightarrow$ $f(z) \in (f(x) - \epsilon_{fx}, f(x) + \epsilon_y)$

thus, $(x - \delta_x, x + \delta_x) \subseteq f^{-1}(V)$. this is true $\forall$ every $x \in f^{-1}(V) \Rightarrow f^{-1}(V)$ is open.

$(\Leftarrow)$ Munkres

common cts functions

① Any function that maps a space w/ the discrete top to a top. space is cts.

$f:$ 

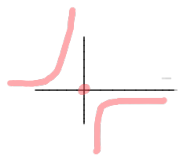
②  $f: D \to \{x\}$

D" the set $\{x\} \subseteq \mathbb{R}^2$ w/ the subspace top. = Target

$\{x\}$ is open

③ non-example $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$



the set $(-1,1) \subset \underbrace{\mathbb{R}}_{\text{Range}}$ is open

$f^{-1}(-1,1) = (-1,0) \cup \{0\} \cup (0,1)$

even though as a set this is all of R, it's not open since it's the union of two open sets and the non-open {0}

④ $f: \mathbb{R} \to \mathbb{R}$ is cts b/c. open set $V \subset \mathbb{R}$, say (a,b)

$f(x) = x^2$

$f^{-1}(a,b)$