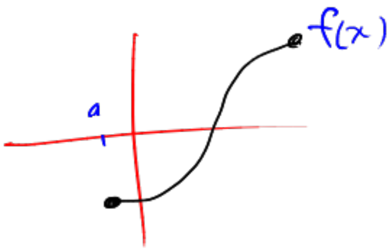


Intermediate Value Theorem

(for polynomials)

($\Rightarrow$ no breaks

(continuous)

If $f(a) < 0$ & $f(b) > 0$

then you're guarenteed a point c between a & b

$$f(c) = 0$$

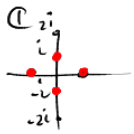
Complex numbers:

- invented to provide solutions to any equation.

  $x^2+4=0$ no real solutions but

  $x^2=-4 \Rightarrow x = \pm\sqrt{-4} = \pm\sqrt{4}\cdot\sqrt{-1} = \pm 2i$ 2 complex (imaginary solutions)

- generally $\underset{\text{real part}}{a} + \underset{\text{imaginary part}}{bi}$

- 

- Multiplication: $(a+bi)(c+di) = ac + adi + cbi + \overset{-bd}{bdi^2}$

  $= (ac-bd) + (ad+bc)i$

  mult. by i

  start $1\cdot i = i$

  $i\cdot i = -1$

  $-1\cdot i = -i$

  $-i\cdot i = -i^2 = -(-1) = 1$ end

- $x^2 + 2x + 3 = 0$ $\sqrt{8}\cdot\sqrt{-1} = \sqrt{8}\,i = (2\sqrt{2})i = 2i\sqrt{2}$

  $x = \dfrac{-2\pm\sqrt{4-4\cdot 3}}{2} = \dfrac{-2\pm\sqrt{-8}}{2} = \dfrac{-2\pm 2i\sqrt{2}}{2} = -1\pm i\sqrt{2}$

  factors into $(x-(1+i\sqrt{2}))(x-(1-i\sqrt{2}))$

- $x^2+1=0$ doesn't factor (over real #'s)

  $(x-i)(x+i) = x^2 - ix + ix - i^2 = x^2+1$ ✓

- Division: $\dfrac{1+2i}{3+4i}\cdot\dfrac{3-4i}{3-4i} = \dfrac{3-4i+6i-8i^2}{9+16} = \dfrac{11+2i}{25}$

  $= \left(\frac{11}{25}\right) + \left(\frac{2}{25}\right)i$

- cute tricks: compute $i^4 =$

  $i^{100} =$

  $i^{101} =$

  $i^{103} =$

- Factor Completely $3x^5 + 24x^3 + 48x$

  $3x(x^4 + 8x^2 + 16)$

  $3x(x^2+4)^2$

  $3x((x-2i)(x+2i))^2$

  $3x(x-2i)^2(x+2i)^2$

$$f(x) = \frac{-8x^4+56x^3-7x^2+4x+8}{1x^4-7x^3-5x^2+7x+8} \approx -\frac{8x^4}{x^4} \text{ when } x \text{ is LARGE}$$

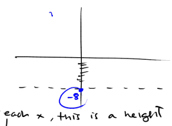
$$" -8$$

$$f(x) = \frac{\left[-8x^4+56x^3-7x^2+4x+8\right]\left(\frac{1}{x^4}\right)}{\left[1x^4-7x^3-5x^2+7x+8\right]\left(\frac{1}{x^4}\right)} = \frac{-\frac{8x^4}{x^4}+\frac{56x^3}{x^4}-\frac{7x^2}{x^4}+\frac{4x}{x^4}+\frac{8}{x^4}}{\frac{1x^4}{x^4}-\frac{7x^3}{x^4}-\frac{5x^2}{x^4}+\frac{7x}{x^4}+\frac{8}{x^4}}$$

$$= \frac{-8+\frac{56}{x}-\frac{7}{x^2}+\frac{4}{x^3}+\boxed{\frac{8}{x^4}}}{1-\frac{7}{x}-\frac{5}{x^2}+\frac{7}{x^3}+\frac{8}{x^4}}$$

now let $x \to \infty$

$= -8$



for each x, this is a height

solutions to this are points with height $= -8$. (where the height is achieved)

$$f(x) = \frac{-8x^4+56x^3-7x^2+4x+8}{1x^4-7x^3-5x^2+7x+8} = -8$$

cross multiply

$$\cancel{-8x^4}+\cancel{56x^3}-7x^2+4x+8 = \cancel{-8x^4}+\cancel{56x^3}+40x^2-56x-64$$

Quadratic Formula

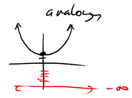
$$0 = 47x^2-60x-72$$

$$x = \frac{+60 \pm\sqrt{60^2-4(47)(-72)}}{2(47)}$$

real

$=$

analogy



x^2+2

if we set $= -8$

$x^2+2=-8$

$x^2=-10$

$x = \pm\sqrt{-10}$

no real

$$P(x) = 4x^4-8x^3-8x^2-8x-12$$

Rational Zeros:

Possible Rational Zeros: $\pm \frac{\text{Factors of } 12}{\text{Factors of } 4} = \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{6}{1}, \frac{12}{1}$

$\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{6}{2}, \frac{12}{2}$

$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{6}{4}, \frac{12}{4}$

$x-(-1) = x+1 \longrightarrow$ guarantee factor

$$\boxed{P(-1)} = 4+8-\underbrace{8+8}_{0}-12 = \boxed{0}$$

$= -8$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}$

start here

when you get $P(x)=0$, long divide to find other factors.

long divide ↓ to find other roots

repeat this sub

$$\begin{array}{r} \boxed{4x^3-12x^2+4x-12} \\ x+1 \overline{) 4x^4-8x^3-8x^2-8x-12} \\ 4x^4+4x^3 \\ \hline -12x^3-8x^2-8x-12 \\ -(-12x^3-12x^2) \\ \hline 4x^2-8x-12 \\ -4x^2+4x \\ \hline -12x-12 \\ -12x-12 \\ \hline 0 \end{array}$$

now we want zeros of this / or factors.

$q(x) = 4x^3-12x^2+4x-12$

Table — find zeros.

$$x^2+1=0$$

no real solutions

$x^2=-1$

300+ years ago, complex #'s

$x = \pm\sqrt{-1}$

$\longleftrightarrow$ $\mathbb{R}$ add $\sqrt{-1} = i$

$x = \pm i$

$\boxed{-1 = \sqrt{-1}\sqrt{-1} = i^2}$

General Complex #: $\boxed{a+bi}$

a is real

b is real.

multiply just like binomials: $(1+2i)(3+4i) = 1\cdot3+4i+\underbrace{3\cdot2i}_{6i}+8i^2$ $\quad i^2=-1$

$= 3+10i-8 = -5+10i$

$8i^2 = 8(-1) = -8$

Divide Complex #'s

$$\frac{4+8i}{1+2i}\left(\frac{1-2i}{1-2i}\right) = \frac{4-8i+8i+16}{1+4} = \frac{20}{5} = 4$$

$$\frac{4+8i}{3+5i}\left(\frac{3-5i}{3-5i}\right) = \frac{12-20i+24i+40}{9+25} = \frac{52+4i}{36}$$

same coeffs - signs opposite

always real down stairs

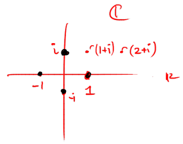
now separate $= \boxed{\frac{52}{36}+\frac{4}{36}i}$

a b

You can always factor over complex #s $\mathbb{C}$

$x^2+1 = (x-i)(x+i)$

$\mathbb{C}$



start w/ repeatedly multiply by i

plot result each time

$1 \cdot i = i$

$i \cdot i = -1$

$-1 \cdot i = -i$

$-i \cdot i = 1$

$i^4 = 1$

$i^{100} = 1$

$i^{103} = \underline{-i}$

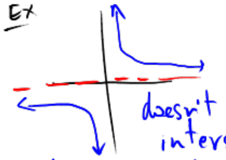
multiply bases add exponents

$i^{103} = i^{100} \cdot i^3$

$" 1 \cdot i^3 = -i$

1. Is it possible for a graph to:
(a) intersect its vertical asymptote? No, since V. Asy $\Leftrightarrow$ division by 0
(b) " " horizontal asymptote? Sometimes.

Ex



$1/x$

doesn't intersect horizontal asy.

set/solve

$x\left(\frac{1}{x}\right) = 0\,(x)$

$1 = 0$

nonsense

Ex. $f(x) = \dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

Horizontal Asy: $\boxed{y=1}$

behavior as $x \to \infty$

$\approx \dfrac{x^2}{x^2}$ when $x \to \infty$

$= 1$

Does graph of f(x) intersect its Horizontal Asymptote?

set/solve $f(x) = 1 = \dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

cross multiply:

$\Rightarrow (x-3)(x-4) = (x-1)(x-2)$

$\cancel{x^2} - 7x + 12 = \cancel{x^2} - 3x + 2$

$10 = 4x \Rightarrow x = 10/4 = 2.5$ this is the x-value for which $f(x) = 1$

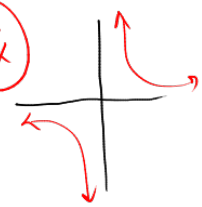
so..

$f(2.5) = 1$

Yes, the graph intersects the horizontal asy.

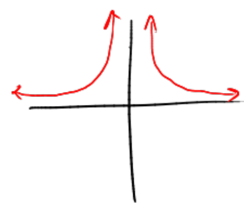
$\boxed{\text{no real sols} \Rightarrow \text{NO}}$

$1/x$

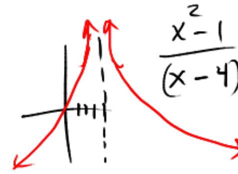


Near vertical asy the graph splits

$f(x) = \dfrac{1}{x^2}$



Here, the graph "converges"



$\dfrac{x^2-1}{(x-4)^2}$

Determine all asymptotes, zeros, graph.

$\dfrac{x^3-8}{(x-1)^2}$

H. Asy / End Behavior

$\approx \dfrac{x^3}{x^2} = x$ growing just like $y = x$

zeros: $x^3 - 8 = 0$

$x^3 = 8,\ x = 2$

V. Asy: $(x-1)^2 = 0$

$x = 1$

$y = x + 2$

slant asy.

Long Division:

$$\begin{array}{r} x + 2 \\ x^2 - 2x + 1 \enclose{longdiv}{x^3 - 8} \\ -(x^3 - 2x^2 + x) \\ 2x^2 - x - 8 \\ -(2x^2 - 4x + 2) \\ 3x - 10 \end{array}$$

$\dfrac{x^3-8}{(x-1)^2} = (x+2) + \boxed{\cancel{\left(\dfrac{3x-10}{x^2-2x+1}\right)}} \to 0$

as $x \to \infty$

$x^2+1=0$

No real solutions
there are complex (or imaginary) solutions.

$x^2=-1$

$x=\pm\sqrt{-1}$

$(x-i)(x+i)$

$\underbrace{(x-i)(x+i)}$

$x^2+ix-ix-i^2$

$\boxed{x^2+1}$

300+ years ago, complex #'s were invented as a way to make sure that ALL equations had solutions

Definition: $\sqrt{-1}=i \Rightarrow \boxed{i^2=-1}$

add it to our real #'s.

complex number:

$a+bi$

a ↳ real #, b ⟶ real #.

Multiply Complex #'s

$(1+2i)(3-4i) = 1\cdot 3 - 4i + 6i$ $\quad \underbrace{(-4i)(2i)}$ $\quad \boxed{-8i^2} = (-8)(-1) = 8$

$= \boxed{11+2i}$

complex conjugate of 3-4i

$+8i^2$

$$\frac{(1+2i)}{(3-4i)}\cdot\frac{(3+4i)}{(3+4i)} = \frac{3+4i+6i-8}{9+16} = \frac{-5+10i}{25}$$

$(2i)(4i)=8i^2=-8$

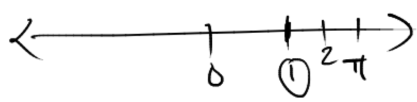
now separate

↳ always real

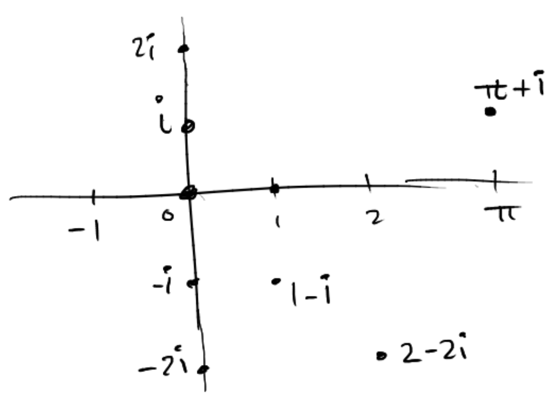
$$=\frac{-5}{25}+\frac{10}{25}i$$

$$=-\frac{1}{5}+\frac{2}{5}i$$

ℝ-line



ℂ-plane



start: 1

multiply by i = i

repeat: $i\cdot i = i^2 = -1$

repeat $-1\cdot i = -i$

repeat: $-i\cdot i = -i^2 = 1$