Long Division of Polynomials

$$\frac{8x^3 - 4x^2 + 7x - 14}{x^3+1} = 8 + \frac{-4x^2+7x-22}{x^3+1}$$

$$\begin{array}{r l}
 & 8 \\
x^3+1 \,\big) & 8x^3 - 4x^2 + 7x - 14 \\
 & -(8x^3 \qquad\qquad\quad + 8) \\
\hline
 & 0 \quad -4x^2 + 7x - 22
\end{array}$$

$\leftarrow$ remainder: degree less than that of the divisor

stop when:

$$\frac{x^3+3x+5}{x+1} \Rightarrow$$

find quotient and remainder:

$$\begin{array}{r l}
 & x^2 - x + 4 \\
x+1 \,\big) & x^3 \qquad + 3x + 5 \\
 & -(x^3 + x^2) \\
\hline
 & -x^2 + 3x + 5 \\
 & -(-x^2 - x) \\
\hline
 & 4x + 5 \\
 & -(4x+4) \\
\hline
 & 1
\end{array}$$

$$f(x) = x^3 - 8x^2 + 19x - 12 = (x-1)(x-3)(x-4)$$

1   3   4

$x=1 \Rightarrow f(1) = 1^3 - 8 + 19 - 12 = 0$

— $x=1$ is a zero of $f(x)$

$\Rightarrow (x-1)$ is a factor of $f(x)$

$$x^2 - 7x + 12 = (x-3)(x-4)$$

$$\begin{array}{r l}
 & x^2 - 7x + 12 \\
x-1 \,\big) & x^3 - 8x^2 + 19x - 12 \\
 & -(x^3 - x^2) \\
\hline
 & -7x^2 + 19x - 12 \\
 & -(-7x^2 + 7x) \\
\hline
 & 12x - 12 \\
 & 12x - 12 \\
\hline
 & 0
\end{array}$$

$P(x) = 2x^4 - 6x^2 - 8 = 0$

$x^4 - 3x^2 - 4 = 0$

$\underbrace{(x^2-4)}_{\pm 2}\underbrace{(x^2+1)}_{\text{complex 0's}} = 0$

$P(x) = x^4 - x^3 - 4x^2 - 2x - 12$ 1, 2, 3, 4, 6, 12

Rational Zeros: divisors of $-12/1 = -12$ are possible:

$P(2) = 16 - 8 - 16 - 4 - 12$

$P(3) = 81 - 27 - 36 - 6 - 12 = 0$

$$\begin{array}{r|l} & x^3 + 2x^2 + 2x + 4 \\ x-3 & x^4 - x^3 - 4x^2 - 2x - 12 \\ & -(x^4 - 3x^3) \\ & \quad 2x^3 - 4x^2 - 2x - 12 \\ & -(2x^3 - 6x^2) \\ & \qquad 2x^2 - 2x - 12 \\ & \qquad 2x^2 - 6x \\ & \qquad\qquad 4x - 12 \end{array}$$

$\longrightarrow$ rational zeros: could be $\pm 1, \pm 2, \pm 2$

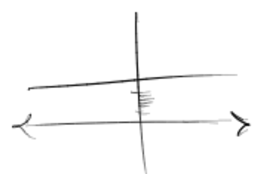
$-1 + 2 - 2 + 4$ NO (-1)

$-8 + 8 - 4 + 4 = 0$ (-2)

$$\begin{array}{r|l} & x^2 + 2 \\ x+2 & x^3 + 2x^2 + 2x + 4 \\ & x^3 + 2x^2 \\ & \qquad 2x + 4 \end{array}$$

so $P(x) = (x-3)(x+2)(x^2+2)$

Horizontal Asymptote: $y = -8$



$f(x) = -8$

$-8x^4 + 56x^3 - 7x^2 + 4x + 8 = -8x^4 + 56x^3 + 40x^2 - 56x - 64$

$33x^2 - 60x - 72 = 0$

$\Rightarrow 11x^2 - 20x - 24 = 0$

quadratic formula to find where

LHS: $x = 0 \Rightarrow$ LHS $= -24$

$x =$ HUGE $\Rightarrow$ LHS > 0

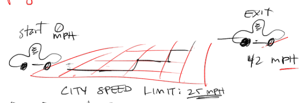
$\{x \to \infty$ forces $y \to \infty\}$

I.V.T $\to$ Real solutions Exist at some point.

Ideas on WeBWorK / Chapter 3

- I.V.T for polys : 2 motivations - 1 real world, travel

1. start



CITY SPEED LIMIT: 25 mph

were you speeding in city?

were you ever traveling 25 mph? 35? → yes Intermediate Value Theorem.

Is there a real zero? Poly ⇒ I.V.T.

Mathematics

$f(x) \to \infty$ as $x \to \infty$ $\quad f(x) \to \infty$ as $x \to -\infty$

$f(x) = 4x^2 + 2x - 1$

$\{g(x) \to -\infty$ as $x \to \infty\}$

$g(x) = -3x^2 + x + 200$

I.V.T ⇒ $g(c) = 0$ for some $c \in (0, 10]$

$h(x) = x^3 - 100x + 10$ $\quad h(1) < 0$

$h(x) \to -\infty$ as $x \to -\infty$

Can you mentally find an x-value which makes LHS negative? Positive? (End Behavior analysis will help)

If ∞ — you're guaranteed this quadratic has ① .10 +

End Behavior:

$P(x) = x^3 + \cancel{\boxed{100x - 1000}}$ think: x very large (1000, 1 million, etc) only leading term matters $(1000)^3$

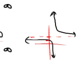
$P(x) \to \underline{\infty}$ if $\boxed{x \to \infty}$

$P(x) \to \underline{-\infty}$ if $x \to -\infty$

behaves like $1/x$ as x gets BIG $\quad x^3/x^4 = 1/x \quad$ $1/1000000 = .000001 \approx 0$

$f(x) = \dfrac{x^3 + 100x + 1}{x^4 + 2}$ $\quad f(x) \to \underline{0}$ as $x \to \infty$, $\quad f(x) \to \underline{0}$ as $x \to -\infty$

$f(x) = \dfrac{2x^3 + 1}{x^3 + 2x} \approx 2$ $\quad f(x) \to \underline{2}$ as $x \to \infty$, $\quad f(x) \to \underline{2}$ as $x \to -\infty$



$\boxed{f(x) = \dfrac{3x^3 + 2x}{x + 1}}$ $\quad \dfrac{3x^3}{x} = 3x^2$ $\quad f(x) \to \underline{\infty}$ as $x \to \infty$, $\quad f(x) \to \underline{\infty}$ as $x \to -\infty$

zeros ⟺ factors (Factor Theorem) $f(x) = x^2 - 7x + 12$

$f(4) = 16 - 28 + 12 = 0$, $f(3) = 9 - 21 + 12 = 0$

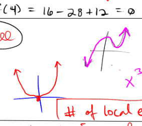
$(x-4)(x-3)$

local minima / maxima vs degree

$8x^3 + 4x^2 - 8x = 4x(2x^2 + x - 2)$

$=$

$(x-1)^4 =$



x^3 — degree $= n$

# of local extrema $\le n-1$

Factor degree 3/4 polys into linear/irreducible factor

Find rational zeros of poly

Horizontal Asymptotes

FACTOR THM:

$$P(x) = x^2 - 7x + 12 = (x-4)(x-3)$$

$$P(4) = 16 - 28 + 12 = 0$$

$$P(3) = 9 - 21 + 12 = 0$$

⇒ zeros: 3, -11, 4

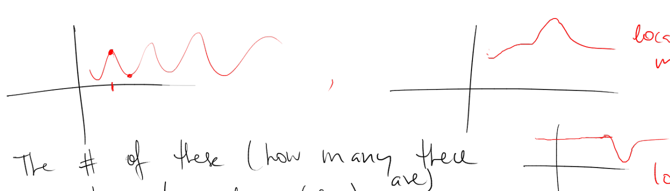
⑦ $f(x) = x^3 + 4x^2 - 65x + 132 = (x-3)(x^2 + 7x - 44) = (x-3)(x+11)(x-4)$

$f(3) = 0$ means $x - 3$ is a factor, $f(3) = \underbrace{27 + 36}_{63} \underbrace{- 195 + 132}_{-63} = 0$

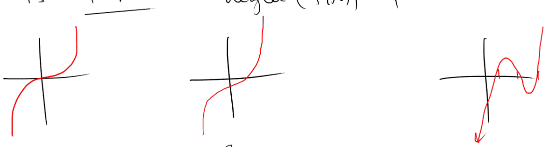
$$\begin{array}{r} x^2 + 7x - 44 \\ x - 3 \overline{) x^3 + 4x^2 - 65x + 132} \\ -(x^3 - 3x^2) \\ \hline 7x^2 \\ -(7x^2 - 21x) \\ \hline -44x \\ -44x + 132 \\ \hline 0 \end{array}$$

What are all the zeros of $f(x)$?

LOCAL EXTREMA or local max / local min

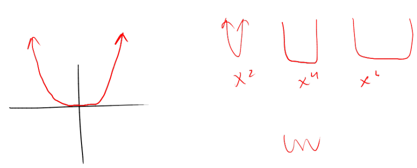


The # of these (how many there are) is at most degree$(f(x)) - 1$



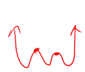
x^3 $\quad$ $x^3 - 1$ $\quad$ $(x-1)(x-2)(x-3)$

no local max/min



$f(x) = (x-1)^4$

$(x-1)(x-2)(x-3)(x-4) \longrightarrow$



$2 \le 4 - 1 = 3$

MA115 :: Sections 3.1 & 3.2 & 3.3 Polynomials, Long Division & Rational Zeros

1. The leading term governs the *end behavior*. Describe the end b ehavior for these functions $f(x) = x^2, f(x) = x^2 + x, f(x) = -x^3, f(x) = -x^3 + 10x^2$, $h(x) = 1/x$, $g(x) = \frac{-2x^2+x+1}{x+1}$

$f(x)$	$f(x) \to \square$ as $x \to \infty$	$f(x) \to \square$ as $x \to -\infty$	
x^2	∞	∞	
$x^2 + \cancel{x}$	∞	∞	
$-x^3 + 10x^2$	$-\infty$	$-(-x)^3 \to \infty$	
$1/x$	$1/\text{BIG} \to 0$	$1/-\text{BIG} \to 0$	
•	$-\infty$	∞	

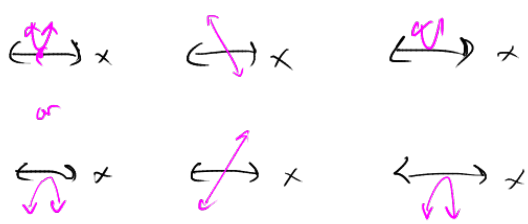
$\frac{-2x^2+x+1}{x+1} \approx -2x^2/x = -2x$

2. Graph by hand $P(x) = x^6(x-3)^5(x+2)^2$. Check your answer by plotting using a graphing utility using a window $-1 \le x \le 3.5$.

Zeros: 0, 3, −2

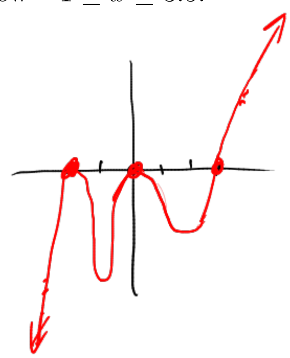
multiplicity: 6 even, 5 odd, 2 even

Shape!:



or

kiss pierce



End Behavior

Total degree: $6+5+2 = 13$ odd

coef of leading term $= 1 > 0$

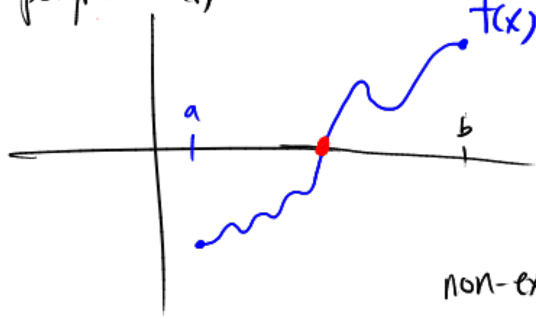
Intermediate Value Theorem (for polynomials)

if $f(a) < 0$

$f(b) > 0$

you're guaranteed a ~~point~~ number c between a & b where

$$f(c) = 0$$



non-example

Does $f(x)$ have a real zero?

$f(x) = x^2 + x - 1$ yes

$f(0) = -1 < 0$

$f(3) > 0$ I.V.T. $\Rightarrow$ there's a # c in $(0,3)$ where $f(c) = 0$

End Behavior:

$g(x) = -x^2 + 100x + 10$

$g(x) \longrightarrow -\infty$ as $x \longrightarrow \infty$ (you can eventually find a s.t. $f(a) < 0$

$f(0) > 0 \Rightarrow c$ in $(a, 0)$

$h(x) = x^3 + 10x + 1$

↳ Yes, b/c as $x \to -\infty$, $h(x) \longrightarrow (-x)^3 + 10(-x) + 1$

$= -x^3 - 10x + 1 \longrightarrow -\infty$

Complex Numbers & Rational Functions

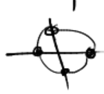
$i = \sqrt{-1}$

Every poly has root in $\mathbb{C}$

- when encountered, include as solutions — don't just say no solutions

- know how to multiply
  $(1+i)(1-i) = 1 - i^2 = 2$

- Multiplication by i is rotation



$i^{103} = i^{100} \cdot i^3 = i^3 = -i$

$(i^4)^{25}$

$i^4 = 1 \Rightarrow i^{100} = 1$

Horizontal Asymptotes
Vertical Asymptotes (where denom $= 0$)

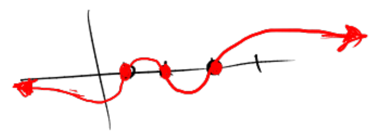
graph "approaches" asymptote.
— sometimes intersecting it

$x \to \infty$
$\frac{x^3}{x^4} = \frac{1}{x} > 0$
$x \to -\infty < 0$

$$\frac{(x-1)(x-2)(x-3)}{x^4+1} = f(x)$$

Crosses Horizontal Asy.

$\to 0$ as $x \to \infty$



$$f(x) = \frac{x^2 - 4x - 45}{x^2 - 9}$$

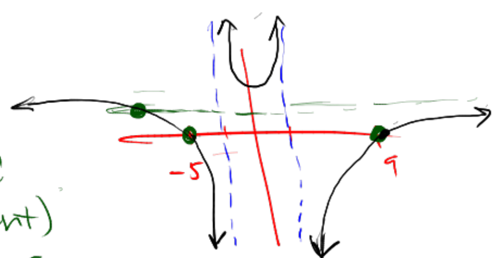
Domain / Vertical Asy — Domain — not this — $x = \pm 3 \to$ vertical asy

Zeros

Horizontal Asy

$f(x) \to \underline{\ 1\ }$ as $x \to \pm\infty$

Does it cross Horiz Asy?
Solve $f(x) = 1$ (Height)
$\cancel{x^2} - 4x - 45 = \cancel{x^2} - 9$
$-4x = 36$
$x = -9$



check
$f(0) \geq 0$
$f(\cancel{2.9}\ 3.1) \approx f(3) \approx \frac{-48}{+.001}$
> 0

$f(x) = x^3 + 4x^2 - 65x + 132 = (x-3)(x+11)(x-4) \Rightarrow$ zeros:
$x=3$
$x=-11$
$x=4$

$f(3)=0$ tells us that $x-3$ is a factor of this

If we long divide $x-3$ into $f(x)$ we'll find the other factors.

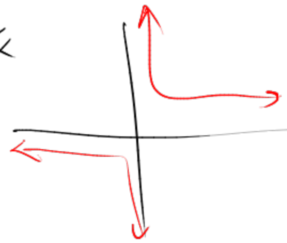
$$
\begin{array}{r}
x^2 + 7x - 44 = (x+11)(x-4) \\
x-3 \overline{\big) x^3 + 4x^2 - 65x + 132} \\
-(x^3 - 3x^2) \\
\hline
7x^2 - 65x + 132 \\
-(7x^2 - 21x) \\
\hline
-44x + 132 \\
-44x + 132 \\
\hline
0
\end{array}
$$

$$\underline{f(3)} = \underbrace{27 + 36}_{\cancel{69}\ 63} - \underbrace{\overset{-195}{65(3)} + 132}_{-63}$$

$$= \cancel{-72}\ 0$$

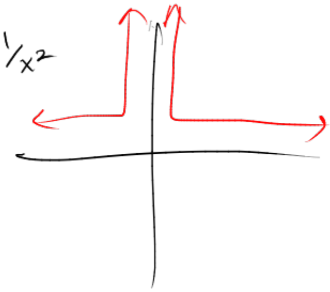
Rational Functions

$1/x$



vs

$1/x^2$



Ex. $f(x) = \dfrac{2x^2 - 4x + 5}{(x-1)^2}$

Vertical Asy: $x = 1$

Hor. Asy: $f(x) \to \boxed{2}$ as $x \to \infty$

Horizontal Asy

Check Points

$x = 1.1$

$f(x) \approx \dfrac{2-4+5}{.1^2} > 0$

$x = .9$

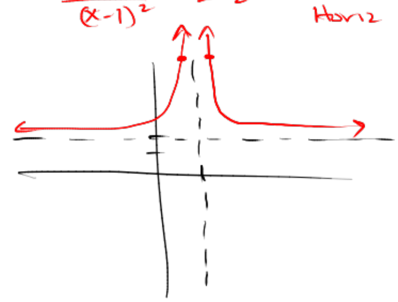
$f(x) \approx \dfrac{2-4+5}{(-.1)^2} > 0$

$\dfrac{2x^2-4x+5}{(x-1)^2} = 2$

$\Rightarrow 2x^2-4x+5 = 2x^2-4x+2$

$\Rightarrow 5 = 2$ ⊗

Never Intersects Horiz Asy



Ex $\dfrac{(x-1)(x-2)}{(x-3)(x+4)}$

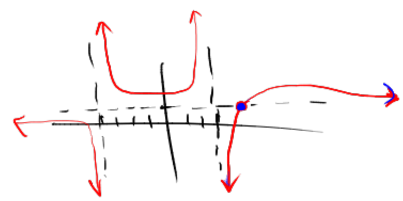
$x = 10 \Rightarrow \dfrac{+ \cdot +}{+ \cdot +} = +$

H.A $= 1$

$x^2 - 3x + 2 = x^2 + x - 12$

$\Rightarrow 14 = 4x$

$3.5 = \frac{7}{2} = x$



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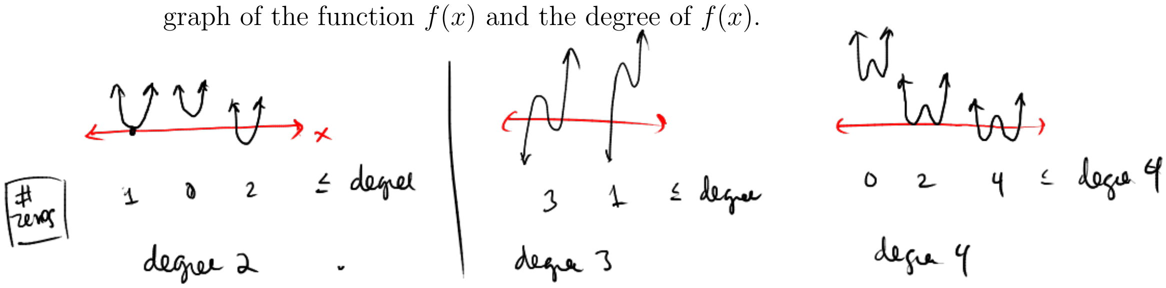
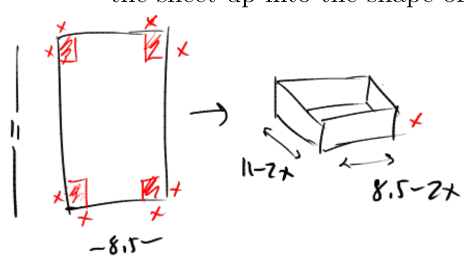
2. Graph by hand $P(x) = x^6(x-3)^5(x+2)^2$. Check your answer by plotting using a graphing utility using a window $-1 \leq x \leq 3.5$.

3. What is the relationship between the maximum number of zeros of the graph of the function $f(x)$ and the degree of $f(x)$.



4. Cut out squares from each corner from a sheet of paper, and then fold the sheet up into the shape of an open box.



$V = x(11-2x)(8.5-2x)$

What size of square should you cut out to create the largest (cm volume) box possible?

Are there two different ways to make a box of volume 40 cubic cmches?

(x^2-x-2)

$12[(x-2)(x+1)(x-3)]$

5. Find a polynomial of degree 3 that has zeros $2, -1, 3$ and whose x coefficient is 12. Hint: zeros $\leftrightarrow$ factors

$$P(x) = 12[x(x^2-x-2) - 3(x^2-x-2)]$$

$$= x^3 - x^2 - 2x - 3x^2 + 3x + 6 = 12[x^3 - 4x^2 + x + 6]$$

want 12

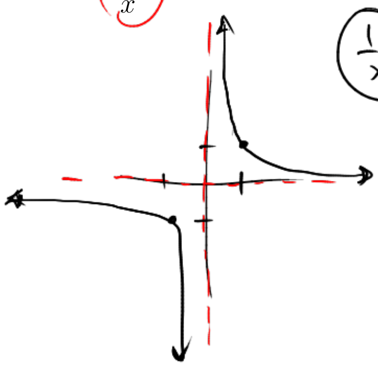
$$Q(x) = 12x^3 - 48x^2 + 12x + 72$$

Same zeros ...... we simply vertically stretched the graph.

$\frac{1}{x} = 0$

6. $\frac{1}{x}$ is a rational function with NO zeros. Sketch a graph of it.

$\frac{1}{x}$ hyperbola



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7. Suppose we happen to know that $f(3) = 0$ where

$$f(x) = x^3 + 4x^2 - 65x + 132$$

. Long division can help us factor and find the remaining zeros.

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$1 + 3 - 49 + 45 = 0$ ✓

8. The function $f(x) = x^3 + 3x^2 - 49x + 45$ has one zero at $x = 1$. Find the remaining zeros.

$x-1$ is a factor of $f(x)$

$$
\begin{array}{r|l}
 & x^2 + 4x - 45 \\
x-1 & x^3 + 3x^2 - 49x + 45 \\
 & -(x^3 - x^2) \\
 & \quad 4x^2 - 49x + 45 \\
 & \quad -(4x^2 - 4x) \\
 & \qquad -45x + 45 \\
 & \qquad -45x + 45 \\
 & \qquad\qquad 0
\end{array}
$$

$\Rightarrow f(x) = (x-1)(x^2+4x-45)$

$= (x-1)(x+9)(x-5)$

zeros: $1, -9, 5$

9. Long division can be useful for relating new functions to old ones.

$$h(x) = \frac{3x+5}{x+2} \text{ is similar to } \frac{1}{x}$$

Show that the graph of h(x) is obtained by tranforming the graph of 1/x.

ALWAYS TRUE!

$$
\begin{array}{r|l}
 & 3 \\
x+2 & 3x+5 \\
 & -(3x+6) \\
 & \quad -1
\end{array}
$$

$$\frac{3x+5}{x+2} = \underbrace{3}_{\text{quotient}} + \frac{\overset{\text{remainder}}{-1}}{\underset{\text{divisor}}{x+2}}$$