

MA 163 Exam 3

1. Find the third degree Taylor polynomial for $x^{3/2}$ about $x = 1$.

$$f(x) = x^{3/2} \qquad f(1) = 1$$

$$f'(x) = \frac{3}{2}x^{1/2} \qquad f'(1) = \frac{3}{2}$$

$$f''(x) = \frac{3}{4}x^{-1/2} \qquad f''(1) = \frac{3}{4}$$

$$f^{(3)}(x) = -\frac{3}{8}x^{-3/2} \qquad f^{(3)}(1) = -\frac{3}{8}$$

Using the above calculations (and don't forget the factorials) . . .

$$x^{3/2} \approx 1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{1}{16}(x-1)^3$$

2. Find the arc length of the curve

$$y = \frac{2}{3}x^{\frac{3}{2}} \text{ from } 0 \text{ to } 3$$

Since the derivative is just $\sqrt{x}$

$$l = \int_0^3 \sqrt{1 + (f'(x))^2} \, dx = \int_0^3 \sqrt{1 + x} \, dx = \int_1^4 \sqrt{u} \, dx$$

$$= \frac{2u^{3/2}}{3} \Big|_1^4 = \frac{2}{3}(4^{3/2} - 1^{3/2}) = \frac{2}{3}(8 - 1) = \frac{14}{3}$$

3. The Maclaurin series for $\cos x$ is below.

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Find the interval of convergence.

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{\frac{|x|^{2(n+1)}}{(2(n+1))!}}{\frac{|x|^{2n}}{(2n)!}} = \frac{(2n)!}{(2n+2)!} \cdot \frac{|x|^{2n+2}}{|x|^{2n}} \\ &= \frac{(2n)!}{(2n+2)(2n+1)(2n)!} |x|^2 = \frac{1}{(2n+2)(2n+1)} |x|^2 \\ \lim_{n \rightarrow +\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow +\infty} \frac{1}{(2n+2)(2n+1)} |x|^2 \\ &= |x|^2 \lim_{n \rightarrow +\infty} \frac{1}{(2n+2)(2n+1)} = |x|^2 \cdot 0 = 0 < 1 \end{aligned}$$

By the Ratio Test, this series converges for all values of x .

4. Use the series above to obtain an estimate for $\cos\left(\frac{1}{2}\right)$ to within 0.01 of the actual value.

$$\cos\left(\frac{1}{2}\right) = 1 - \frac{1}{2} \left(\frac{1}{2}\right)^2 + \frac{1}{24} \left(\frac{1}{2}\right)^4 \simeq 0.8776$$

(Alternating series, and the last fraction is less than 0.01, so we can “stop” here.)

5. Use an eighth degree Taylor polynomial to estimate

$$\begin{aligned} \int_0^1 \frac{\cos(x^2) - 1}{x} dx &\simeq \int_0^1 -\frac{1}{2}x^3 + \frac{1}{24}x^7 dx = \left[-\frac{1}{8}x^4 + \frac{1}{192}x^8\right]_0^1 = -\frac{1}{8} + \frac{1}{192} = -\frac{23}{192} \\ \left(\cos(x^2) \simeq 1 - \frac{1}{2}(x^2)^2 + \frac{1}{24}(x^2)^4 = 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 \implies \frac{\cos(x^2) - 1}{x} \simeq -\frac{1}{2}x^3 + \frac{1}{24}x^7\right) \end{aligned}$$

6. The Maclaurin series for the function $\ln(x+1)$ is below. Find the interval of convergence.

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left| \frac{(-1)^{n+1+1}}{n+1} x^{n+1} \right|}{\left| \frac{(-1)^n}{n} x^n \right|} = \frac{n}{n+1} \cdot \frac{|x|^{n+1}}{|x|^n} = \frac{n}{n+1} |x|$$

$$\lim_{n \rightarrow +\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow +\infty} \frac{n}{n+1} |x| = |x| \lim_{n \rightarrow +\infty} \frac{n}{n+1} = |x| \cdot 1 = |x|$$

From the Ratio Test, we know that the series converges for $-1 < x < 1$ and diverges for $(-\infty, -1) \cup (1, +\infty)$. We have to check whether or not the series converges for $x = 1$ and $x = -1$.

When $x = 1$, the power series is equal to

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \text{ which converges}$$

When $x = -1$, the power series is equal to

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots \text{ which diverges}$$

Therefore, the interval of convergence for the Maclaurin Series for $\ln(x+1)$ is $(-1, 1]$.

7. Use # 8 to find a series that converges to $\ln 2$.

$$\text{Plug } x = 1 \text{ into the series in \#8: } \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

8. Prove $e^{i\pi} + 1 = 0$ We know

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

so

$$e^{i\pi} = \sum_{n=0}^{+\infty} \frac{(i\pi)^n}{n!} = 1 + \frac{i\pi}{1} + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!} + \frac{(i\pi)^4}{4!} + \frac{(i\pi)^5}{5!} + \frac{(i\pi)^6}{6!} + \dots$$

since the series converges we can rearrange the order of the terms, keeping the even powers separate from the odd

$$e^{i\pi} = \sum_{n=0}^{+\infty} \frac{(i\pi)^n}{n!} = 1 + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^4}{4!} + \frac{(i\pi)^6}{6!} + \dots + \frac{i\pi}{1} + \frac{(i\pi)^3}{3!} + \frac{(i\pi)^5}{5!} + \dots$$

using the fact that $i^2 = -1$ and $i^4 = 1$ we get

$$e^{i\pi} = \sum_{n=0}^{+\infty} \frac{(i\pi)^n}{n!} = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots + \frac{i\pi}{1} - \frac{i(\pi)^3}{3!} + \frac{i(\pi)^5}{5!} + \dots$$

factoring out i in the second half

$$e^{i\pi} = \sum_{n=0}^{+\infty} \frac{(i\pi)^n}{n!} = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots + i \left(\frac{\pi}{1} - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} + \dots \right)$$

recognizing the odd powers are Maclaurin series for cosine evaluated at $x = i\pi$ and likewise for the odds and sine.

$$e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1 + 0 = -1$$

thus

$$e^{i\pi} + 1 = 0$$