wk 10 Fri

- Exam Guide Sols posted this p.m.
- Exam 3 = Wed!

Review: Estimate $\ln(1.3)$, using Taylor Series @ $a=1$, to within 10^{-3} accuracy (use Taylor's Rem. Thm.

$f(x) = \ln(x)$ | $f(a) = \ln(1) = 0$.

$f'(x) = \frac{1}{x}$ | $f'(1) = 1$

$f''(x) = \frac{-1}{x^2} = -x^{-2}$ | $f''(1) = -1$

$f'''(x) = \frac{2}{x^3} = 2x^{-3}$ | $f'''(1) = 2$

$f^{(iv)}(x) = \frac{-6}{x^4}$ | $f^{iv}(1) = -6$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$

$\downarrow$

$f^{(n+1)}(x) = \frac{(-1)^n n!}{x^{n+1}}$ | $f^{(n+1)}(1) = (-1)^n n!$

$(-1)^{n+1+1} = (-1)^{n+2} = (-1)^n$

$f(x) = \sum \frac{f^n(a)}{n!} x^n$

$R_n(x) \leq \left| \frac{M \cdot |x-1|^{n+1}}{(n+1)!} \right|$

$M \geq f^{n+1}(u) = n!$

So

$\ln(x) = \sum \frac{(-1)^n (n-1)!}{n!} x^n$

$\frac{(n-1)!}{n!} = \frac{1}{n}$

$\ln(x) \approx 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ (A)

How far to go?

$\frac{M \cdot |x-1|^{n+1}}{(n+1)!} = \left| \frac{n! \, |x-1|^{n+1}}{(n+1)!} \right| = \left| \frac{0.3^{n+1}}{n+1} \right| < 10^{-3}$

sub $x = 1.3$

$\left(\frac{3}{10}\right)^{n+1} \cdot 10^3 < n+1$

($n = 4$ works!)

$\left(\frac{3}{10}\right)^5 \cdot 10^3 = \frac{3^5}{10^2} < 5$

$= \frac{243}{100} = 2.4 < 5$

Restrict to an interval away from 0, so choose an interval that includes 1.3 — $[1, \infty)$

$f^{n+1}(x)$ has a max @ $x = 1$

choose $M = \frac{n!}{1^{n+1}} = n!$

sub $x = 1.3$ into (A) go out to $n = 4$

$(1.3) - \frac{(1.3)^2}{2} + \frac{1.3^3}{3} - \frac{1.3^4}{4}$

Approx. $\cos(1.4)$ to within 10^{-3}

| Note: study guide should read: $\sin(1.4)$ use Taylor @ $a=1$

$f(x) = \cos x$ | $f'(1) = \cos(1)$

~~$f' = -\sin x$~~ | ~~$f'(1) = -\sin(1) =$~~ 0

$f'' = -\cos x$ | $f''(1) = -\cos(1)$

~~$f''' = \sin x$~~ | ~~$f'''(1) = 0$~~

$f^4 = \cos x \ldots$ | $f^{4}(1) = \cos(1)$

$$f^n(x) = \begin{cases} \pm \sin x & n \text{ odd} \\ \pm \cos x & n \text{ odd} \end{cases}$$

Choose M

① since $\sin x, \cos x$ are bounded b/w $(-1,1)$ everywhere we don't need to restrict choosing M is easy.

Chose $M = 1$.

② $$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

$$1 \cdot \frac{|x-a|^{n+1}}{(n+1)!} \overset{\text{sub } x=1.4,\ a=1}{=} \frac{|0.4|^{n+1}}{(n+1)!} \leq 10^{-3}$$

③ solve the ineq

$$\left(\frac{4}{10}\right)^{n+1} \cdot 10^3 \leq (n+1)!$$

$$\frac{4^4}{10^4} \cdot 10^3 \overset{?}{\leq} 4! = 24$$

$$= 25.6 = \frac{256}{10}$$

try $n=3$

$4^5 = 2^{2^5} = 2^{10}$ $n=4$?

$$\left(\frac{4}{10}\right)^5 \cdot 10^3 \overset{?}{\leq} 5!$$

$$\frac{1024}{10^5} \cdot 10^3 = \frac{1024}{100} = 10.24 \leq 5!$$ ☺

Taylor Series for $\cos x$ $$\sum \frac{(-1)^n (x-a)^{2n}}{(2n)!} = 1 + \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots$$

sub $1.4 = x$ up to $n=4$ $$\Rightarrow \left(1 + \frac{(1.4)^2}{2} - \frac{(1.4)^4}{4!}\right)$$