MA163 Wk 12 thur

Warm-up:

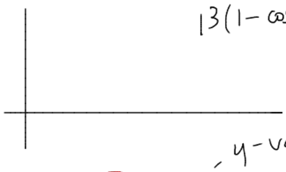
Exercise on Surface Area of a Surface of Rev., given by a parametric curve.

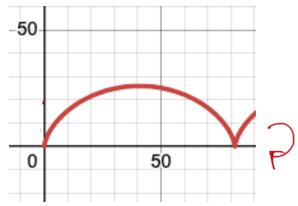
The curve:

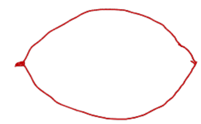
$$\alpha(t) = (13(t-\sin(t)), 13(1-\cos(t)))$$

$\downarrow d/dt$ $\qquad$ $\downarrow d/dt$

$13(1-\cos t)$







y-value

$$S = \int_0^{2\pi} 2\pi \cdot f(t) \cdot \sqrt{1 - f'(t)}\, dt \quad \leftrightarrow \quad 2\pi \int_0^{2\pi} y(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Arc Length of Parametric

$$\int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$13 \cdot 2\pi \int (1-\cos(t)) \sqrt{(13(1-\cos t))^2 + (13 \sin t)^2}\, dt$$

$$\underbrace{13 \cdot 2\pi \int (1-\cos t) \cdot 13}_{} \sqrt{1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{=1}}\, dt$$

$$\text{same} \cdot 13\sqrt{2}\sqrt{1-\cos t}$$

$$\underbrace{13^2 2\sqrt{2}\pi}_{} \int_0^{2\pi} (1-\cos t)^{3/2}\, dt$$

$$\text{same} \int_0^{2\pi} 8 \sin^3\left(\tfrac{t}{2}\right) dt$$

$$13^2 \cdot 16\sqrt{2}\pi \left[-\cos\left(\tfrac{t}{2}\right) + \frac{\cos^3\left(\tfrac{t}{2}\right)}{3} \right]_0^{2\pi}$$

whew!

$$\sin^2 t = \frac{1-\cos(2t)}{2}$$

$$2 \cdot \sin^2 t = 1 - \cos(2t)$$

$$2\sin^2\left(\tfrac{t}{2}\right) = 1 - \cos t$$

so

$$\left(2\sin^2\left(\tfrac{t}{2}\right)\right)^{\frac{3}{2}} = (1-\cos t)^{3/2}$$

$$8 \sin^3\left(\tfrac{t}{2}\right)$$

$x = t/2$

$$\int \sin^3 x\, dx$$

odd power

$$\int \sin^2 x \underbrace{\sin x\, dx}_{du}$$

$u = \cos x$

$$-\int (1-\cos^2 x)\, du$$

$$-\int 1\, du + \int u^2\, du$$

$$-u + \frac{u^3}{3}$$

$$-\cos x + \frac{\cos^3 x}{3}$$

$$-\cos\left(\tfrac{t}{2}\right) + \frac{\cos^3\left(\tfrac{t}{2}\right)}{3}$$

The area of math (diff. equations) is huge. (weather, physics models)

O.D.E (Calc 2)
ordinary diff. eq

1st Order ODE ← highest derivative shown

2nd Order ODE ... nth Order ODE

1st Order ODE:
- Separable
- Linear
- Neither

Calc 2 these two only (Separable, Linear)

P.D.E (calc 3+)
partial diff. eq

partial → In functions that depend on 2 or more variables the derivative is taken on each variable wrt "partial derivative"

ex

$$f(x,y) = (3x+1)\cdot\cos(y)$$

$\frac{\partial}{\partial x}(f)$ = derivative wrt x

$\frac{\partial}{\partial y}(f)$ = derivative wrt y

Linear 1st Order D.E. : FORM — $|\, ax_1 + bx_2 = c$

$$\boxed{y' + p(x)y = q(x)}$$

ex $y' + 4y = 8$

Philosophy of Sol'n:

$(F \cdot G)' = F'G + FG'$

① LHS looks almost like a derivative of a product

② multiply entire eqn by $u(x)$ $\rightsquigarrow$ $\underbrace{u(x) \cdot y' + u(x)p(x)y}_{\text{is this a derivative of a product ?}} = u(x)q(x)$

③ yes if $u(x)p(x) = u'(x)$ ← this is separable

$p(x) = \dfrac{u'(x)}{u(x)}$ $\qquad \int \frac{du}{u} = \ln|u|$

④ Integrating Factor:

$$\boxed{u(x) = e^{\int P(x)dx}}$$

choose $c = 1$

$\int p(x) = \int \frac{u'(x)}{u(x)} = \ln|u(x)|$

$e^{\int p(x)dx} = |u(x)|$

$\quad = e^{\int p(x)dx + c}$

$+/- \quad c\, e^{\int p(x)dx}$ "rename" $e^c \cdot e^{\int p(x)dx}$

⑤ given $y' + p(x)y = q(x)$

(i) multiply by $e^{\int p(t)}$

$$\underbrace{e^{\int p(x)} y' + e^{\int p(x)} p(x)y}_{} = e^{\int p(x)} q(x)$$

reinterpret →

$$\frac{d}{dx}\left[e^{\int p(x)} \cdot y\right] = e^{\int p(x)dx} q(x)$$

(ii) integrate both sides

$$\int \frac{d}{dx}\left[e^{\int p(x)} \cdot y\right] dx = \int e^{\int p(x)dx} q(x)\, dx$$

(iii)

$$e^{\int p(x)} \cdot y = \int e^{\int p(x)} q(x)\, dx \quad \text{work!}$$

(iv)

$$y = \frac{\int e^{\int p(x)} q(x)\, dx}{e^{\int p(x)}}$$

☺

$$y' + 4y = 8$$

① $P(x) = 4$

$$e^{\int P(x)} = e^{\int 4\,dx} = e^{4x}$$

② $$y'e^{4x} + 4e^{4x}y = 8e^{4x}$$

③ $$\frac{d}{dx}(e^{4x}y) = 8e^{4x}$$

④ $$\int \frac{d}{dx}(e^{4x}y) = e^{4x}y = \int 8e^{4x}\,dx = \frac{8}{4}e^{4x} + c$$

$$y = \frac{2e^{4x} + c}{e^{4x}} = 2 + ce^{-4x}$$

genl soln