$$\int \frac{1}{x\sqrt{x^2-9}}\cdot dx$$

$x = 3\sec\theta$ $\quad dx = 3\sec\theta\tan\theta\, d\theta$

$x^2 = 9\sec^2\theta$

$\therefore \frac{x}{3} = \sec\theta \Rightarrow \sec^{-1}\left(\frac{x}{3}\right) = \theta$

$\sqrt{AB} = \sqrt{A}\cdot\sqrt{B}$

$$= \int \frac{1}{3\sec\theta\sqrt{9\sec^2\theta - 9}}\cdot 3\sec\theta\tan\theta\, d\theta = \int \frac{\tan\theta}{\sqrt{9\sec^2\theta - 9}}\, d\theta$$

$$\sqrt{9(\sec^2\theta - 1)}$$

$$\sqrt{9}\sqrt{\sec^2\theta - 1}$$

$$3\sqrt{\tan^2\theta}$$

$$3\tan\theta$$

$$= \int \frac{\tan\theta}{3\tan\theta}\, d\theta = \frac{1}{3}\int 1\, d\theta = \frac{1}{3}\theta + C$$

isolate θ in O.G substitution

$$= \frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right) + C$$

More trig sub

$$\int \frac{1}{\sqrt{4-x^2}}\,dx$$

$x = 2\sin\theta \rightarrow dx = 2\cos\theta\,d\theta$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

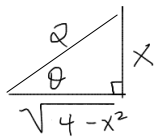
$x^2 = 4\sin^2\theta$

$4 - x^2 = 4 - 4\sin^2\theta = 4(1-\sin^2\theta) = 4\cos^2\theta$

$$\int \frac{1}{\sqrt{4\cos^2\theta}}\,2\cos\theta\,d\theta$$

$$= \int \frac{1}{2\cos\theta}\cdot 2\cos\theta\,d\theta = \int d\theta = \theta + C = \sin^{-1}\left(\frac{x}{2}\right) + C$$

Note: $\sin\theta = \frac{x}{2}$ think $\frac{\text{opp}}{\text{hyp}}$



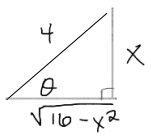
$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{4-x^2}}{2}$

Ex

$$\int \sqrt{16 - x^2}\, dx$$

$x = 4\sin\theta$ $\quad dx = 4\cos\theta\, d\theta$

$x^2 = 16\sin^2\theta$ $\quad \frac{x}{4} = \sin\theta$



$\sqrt{16 - x^2} = \sqrt{16 - 16\sin^2\theta} = \sqrt{16(1 - \sin^2\theta)}$

$= \sqrt{16\cos^2\theta}$

$= 4\cos\theta$

$\sin(A+B) = \sin A\cos B + \sin B\cos A$

$A = B = \theta$ below

$$\int 4\cos\theta\, 4\cos\theta\, d\theta$$

$$= 16\int \cos^2\theta\, d\theta$$

$$= 16\int \frac{1 + \cos 2\theta}{2} = 8\int 1\, d\theta + \frac{1}{2}\, 8\int \cos(2\theta)\, 2\, d\theta$$

$\int \cos(u)\, du = \sin(u)$

$= 4\sin 2\theta$

$u = 2\theta$

$du = 2\, d\theta$

$$= 8\theta + 4\sin 2\theta + C$$

$$= \sin^{-1}\left(\frac{x}{4}\right) + 4(2\sin\theta\cos\theta) + C \qquad 8\cdot\sin^{-1}\left(\frac{x}{4}\right) + 4\cdot 2\cdot\frac{x}{4}\cdot\frac{\sqrt{16 - x^2}}{4} + C$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) + \frac{1}{2}x\sqrt{16 - x^2} + C$$