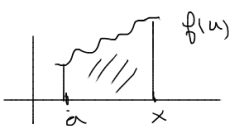


Improper Integrals:

$$\int_a^{\infty} f(x) dx$$

$$\int_{-\infty}^a f(x) dx$$

$$\int_{-\infty}^{\infty} f(x) dx$$

Area so-far function: $\int_a^x f(u) du =$ 

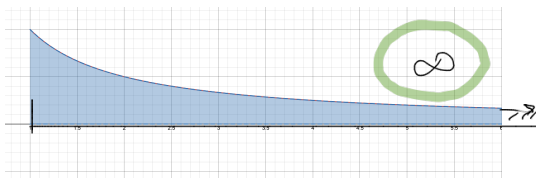
$$\int_a^{\infty} f(x) dx = \lim_{x \rightarrow \infty} \int_a^x f(u) du \stackrel{\text{F.T.C.1}}{=} \lim_{x \rightarrow \infty} (F(x) - F(a)) = \lim_{x \rightarrow \infty} F(x) - F(a)$$

↑ some anti-derivatives

Some examples

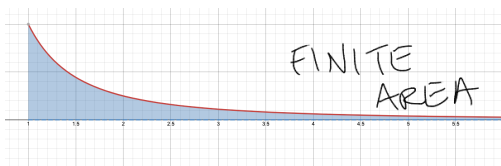
$R > 0, 1 > 0$

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{R \rightarrow \infty} \ln|x| \Big|_1^R = \lim_{R \rightarrow \infty} \ln(R) - \ln(1) = \infty$$



$$\text{Ex } \int_1^{\infty} \frac{1}{\sqrt{x}} dx = \int_1^{\infty} x^{-1/2} = \lim_{R \rightarrow \infty} 2x^{1/2} \Big|_1^R = \lim_{R \rightarrow \infty} 2\sqrt{x} - \lim_{R \rightarrow 1} 1 = \infty$$

$$\text{Ex: } \int_1^{\infty} \frac{1}{x^2} dx = \int_1^{\infty} x^{-2} = \lim_{R \rightarrow \infty} \frac{x^{-1}}{-1} \Big|_1^R = \lim_{R \rightarrow \infty} \frac{-1}{x} \Big|_1^R = \lim_{R \rightarrow \infty} \frac{-1}{R} - \left(\frac{-1}{1} \right) = 1$$



$$\int \frac{1}{x^p} dx \quad \text{"p-integrals"}$$

$$p < 1$$

$$p = 1/4 \quad p = 1/2$$

$$\infty$$

$$p = 1$$

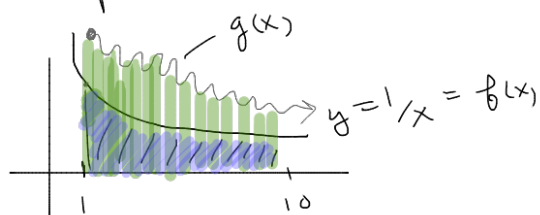
FINITE

$$p = 2$$

$$p = 3$$

$$p > 1$$

Comparison Test



$$\text{If } g(x) \geq f(x)$$

$$\frac{1}{x} \int_1^{\infty} f(x) dx = \infty$$

then

$$\int_1^{\infty} g(x) dx = \infty$$

Ex $\int_1^{\infty} x \sqrt{1 + \frac{1}{x^2}} dx =$ Note: some integrals don't have clean formulas

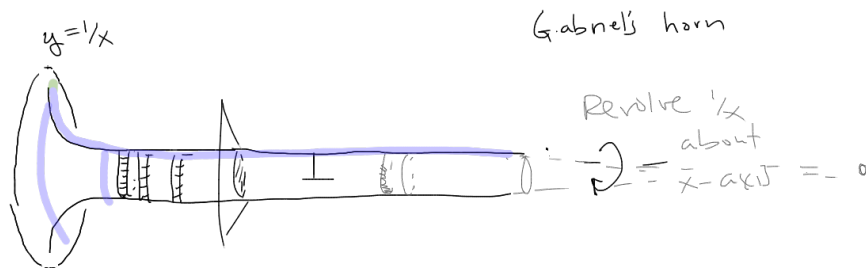
Note: $x \sqrt{1 + \frac{1}{x^2}} > x \sqrt{1 + 0}$

" $g(x)$ " $f(x)$

$$\frac{1}{x} \int_1^{\infty} x dx = \infty \text{ diverges}$$


By comparison test $\Rightarrow \int_1^{\infty} x \sqrt{1 + \frac{1}{x^2}} dx = \infty$ (diverges $= \infty$)

Painter's Paradox



Since the volume of Gabriel's horn is finite, we can fill up the inside entirely with a finite, (pi cubic units) of paint

Volume:

$$\text{slice} = \bigcirc \quad r = 1/x \quad \text{Area} = \pi r^2 = \pi \left(\frac{1}{x^2}\right)$$

$$V = \int_1^{\infty} \text{Area} \, dx = \int_1^{\infty} \pi \left(\frac{1}{x^2}\right) dx = \pi \int_1^{\infty} \frac{1}{x^2} dx = \pi \cdot 1 = \pi$$

Area:

$$f(x) = 1/x$$

$$f' = -1/x^2$$

Surface Area: $\int \text{circumference} \times \text{arc length}$

$$= \int 2\pi r \cdot \sqrt{1 + (f'(x))^2} = 2\pi \int \frac{1}{x} \sqrt{1 + (-1/x^2)^2} dx = 2\pi \int \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx$$

compare

$$\frac{1}{x} \sqrt{1 + \frac{1}{x^4}} > \frac{1}{x} \sqrt{1 + 0} = \frac{1}{x}$$

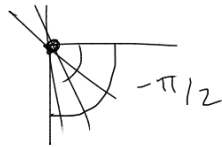
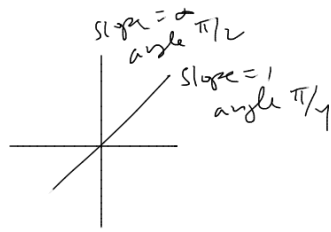
$$\downarrow \int_1^{\infty} \frac{1}{x} dx = \infty \text{ diverges}$$

By comparison test
our integral diverges to $\infty \Rightarrow$

Since the surface is infinite, it takes an infinite amount of paint to cover the outside.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

$$= \lim_{R \rightarrow \infty} \left[\tan^{-1} x \right]_{-R}^R = \lim_{R \rightarrow \infty} \tan^{-1}(R) - \tan^{-1}(-R)$$



$$= \lim_{R \rightarrow \infty} \tan^{-1}(R) - \lim_{R \rightarrow \infty} \tan^{-1}(-R)$$

$$\quad \quad \quad \parallel \quad \quad \quad \parallel$$

$$\quad \quad \quad \frac{\pi}{2} \quad \quad \quad -\frac{\pi}{2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

