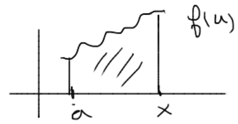
MA163 Wk 4 – Thurs

Improper Integrals : $\int_a^{\infty} f(x)\,dx \qquad \int_{-\infty}^{a} f(x)\,dx \qquad \int_{-\infty}^{\infty} f(x)\,dx$

Area so-far function : $\int_a^x f(u)\,du =$

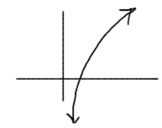


$$\int_a^{\infty} f(x)\,dx = \lim_{x\to\infty} \int_a^x f(u)\,du \overset{\text{F.T.C.}}{=} \lim_{x\to\infty} \big(F(x) - F(a)\big) = \lim_{x\to\infty} F(x) - F(a)$$

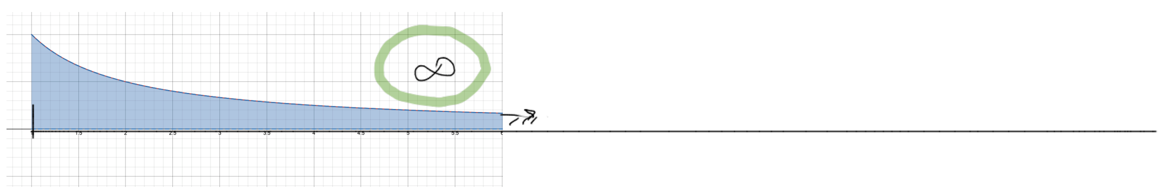
↑ some anti-derivatives

Some examples

$R>0,\ 1>0$



$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{R\to\infty} \ln|x|\Big|_1^R = \lim_{R\to\infty} \ln(x)\Big|_1^R = \lim_{R\to\infty} \ln(R) - \overset{\nearrow 0}{\ln(1)} = \infty$$



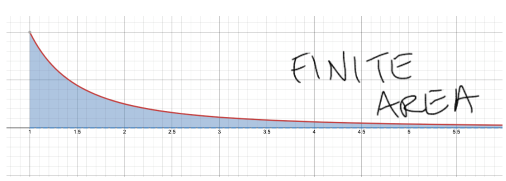
Ex

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \int_1^{\infty} x^{-1/2} = \lim_{R\to\infty} 2x^{1/2}\Big|_1^R = \lim_{R\to\infty} \overset{\nearrow\infty}{2\sqrt{x}} - \lim_{R\to 1} 1 = \infty$$

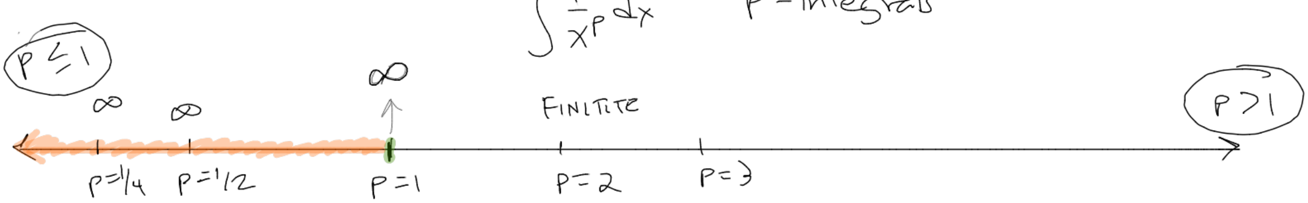
Ex.

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \int_1^{\infty} x^{-2} = \lim_{R\to\infty} \frac{x^{-1}}{-1}\Big|_1^R = \lim_{R\to\infty} \frac{-1}{x}\Big|_1^R = \lim_{R\to\infty} \frac{-1}{R} - \left(\frac{-1}{1}\right) =$$

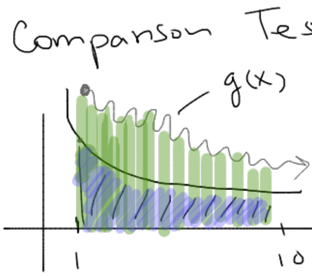
$$= \lim_{R\to\infty} \frac{-1}{R} + 1 = 1$$



$\int \frac{1}{x^p}\,dx$ "p-integrals"



Comparison Test



If $g(x) \geq f(x)$

& $\int_1^\infty f(x)\,dx = \infty$

then

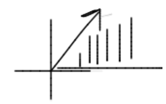
$\int_1^\infty g(x)\,dx = \infty$

Ex $\int_1^\infty x\sqrt{1+\frac{1}{x^2}} =$

Note: some integrals don't have clean formulas

Note: $x\sqrt{1+\frac{1}{x^2}} > x\sqrt{1+0}$

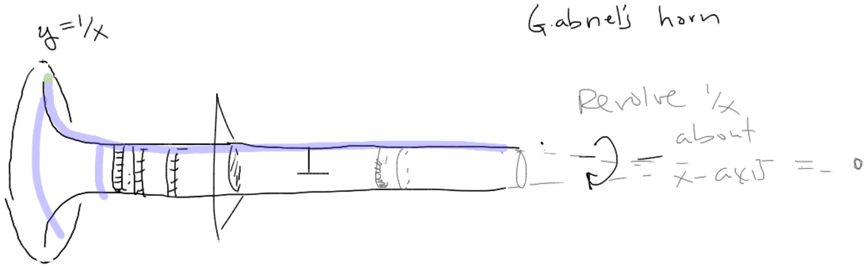
$\quad = g(x) \qquad = f(x)$



& $\int_1^\infty x\,dx = \infty$ diverges

By comparison test $\Rightarrow \int_1^\infty x\sqrt{1+\frac{1}{x^2}}\,dx = \infty$ (diverges $= \infty$)

Painter's Paradox



Since the volume of Gabriel's horn is finite, we can fill up the inside entirely with a finite, (pi cubic units) of paint

Volume:

slice = $r = 1/x$. $\text{Area} = \pi r^2 = \pi\left(\frac{1}{x^2}\right)$

$$V = \int_1^\infty \text{Area}\,dx = \int_1^\infty \pi\left(\frac{1}{x^2}\right)dx = \pi\int_1^\infty \frac{1}{x^2}dx = \pi \cdot 1 = \pi$$

Area:

$f(x) = 1/x$

$f' = -1/x^2$

Surface Area: $\int$ circumference $\times$ arc length

$$= \int 2\pi \overset{1/x}{r} \cdot \sqrt{1 + (f'(x))^2} = 2\pi\int \frac{1}{x}\sqrt{1 + (-1/x^2)^2}\,dx = 2\pi\int \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$$

compare

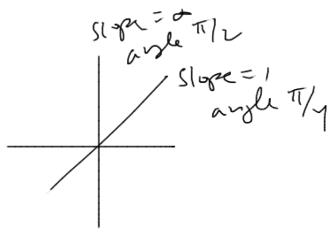
$$\frac{1}{x}\sqrt{1 + \frac{1}{x^4}} > \frac{1}{x}\sqrt{1 + 0} = \frac{1}{x}$$

$\int_1^\infty \frac{1}{x}dx = \infty$ diverges

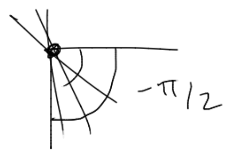
By comparison test
our integral diverges to ∞ $\Rightarrow$

Since the surface is infinite, it takes an infinite amount of paint to cover the outside.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \tan^{-1}x \Big|_{-R}^{R} = \lim_{R\to\infty} \tan^{-1}(R) - \tan^{-1}(-R)$$



$$= \underbrace{\lim_{R\to\infty} \tan^{-1}(R)}_{=\frac{\pi}{2}} - \underbrace{\lim_{R\to\infty} \tan^{-1}(-R)}_{=\frac{-\pi}{2}}$$



$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$