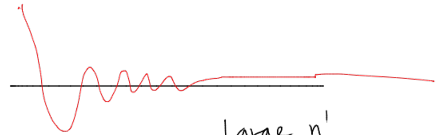
# Root & Ratio Test

**Ratio:** $\sum_{n=1}^{\infty} a_n$ - - - could be positive or negative

If $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} >1 & \text{diverge} \\ <1 & \text{converge} \\ =1 & \text{inconclusive} \end{cases}$$ ✓



large n!

why? Assuming $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$

so $a_{n+1} = r \cdot a_n$ (if $n >> 0$) sufficiently large

and $a_{n+2} = r \cdot a_{n+1} = r^2 a_n$

$a_{n+3} = r \cdot a_{n+2} = r^3 a_n$

$\vdots$

$\forall N$

$\underline{\underline{a_{N+1}}} = \underset{\text{ratio}}{r^N} \cdot \underset{\text{constant}}{a_n}$ $\Leftarrow$ geometric since $r < 1$

the given series is dominated by a convergent geometric series

**ex** $\sum_{n=1}^{\infty} \frac{e^n}{n}$, apply ratio test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$\lim_{n\to\infty} \left| \frac{\left(\frac{e^{n+1}}{n+1}\right)}{\left(\frac{e^n}{n}\right)} \right| = \lim_{n\to\infty} \left| \frac{e^n \cdot e^1}{n+1} \cdot \frac{n}{e^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{e \cdot n}{n+1} \right| \overset{L'H}{=} e > 1$$

— diverges by ratio test —

**ex** $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ $n! \to (n+1)!$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\left(\frac{2^{n+1}}{(n+1)!}\right)}{\left(\frac{2^n}{n!}\right)}$$

converge $0 < 1$

$$\lim_{n\to\infty} \frac{2^{n+1}}{\underset{10!}{(n+1)!}} \cdot \frac{n!}{2^n} = \lim \frac{\cancel{2^n} \cdot 2^1 \cdot \cancel{n!}}{\underset{10}{(n+1)} \underset{9!}{\cancel{n!}} \cancel{2^n}} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

**Factorials**

DEF'n $0! = 1$

$n! = n\cdot(n-1)(n-2)\cdot(n-3)\cdot\ldots\cdot 4\cdot 3\cdot 2\cdot 1$

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

$7! = 7\cdot 6\cdot 5!$

$7!/5! = 7\cdot 6 = 42$

$10! = 10\cdot \underbrace{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}_{9!}$

$= 10\cdot 9!$

**ex** $\lim_{n\to\infty} n! = \infty$

## Root test

$$\sum_{n=1}^{\infty} a_n$$

Q: when to apply?

A: when the term is given as a power of n

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \begin{cases} >1 & \text{diverge} \\ <1 & \text{converge} \\ =1 & \text{inconclusive} \end{cases}$$

ex

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{(n^2+1)^n} = \sum_{n=1}^{\infty} \left(\frac{n+1}{n^2+1}\right)^n \Rightarrow \text{root test}$$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{n^2+1}\right)^n} = \lim_{n\to\infty} \frac{n+1}{n^2+1} \overset{L'H}{=} 0 \qquad \text{converges}$$

ex

For what values does the series converge

$$\sum_{n=1}^{\infty} \frac{7^n}{n^K}$$

let $K$ = constant

say, $K=3$ ? $\sum_{n=1}^{\infty} \frac{7^n}{n^3}$ , ratio.

$$\lim_{n\to\infty} \frac{\left(\frac{7^{n+1}}{(n+1)^3}\right)}{\left(\frac{7^n}{n^3}\right)} = \lim_{n\to\infty} \frac{\cancel{7^n}\cdot 7}{(n+1)^3} \cdot \frac{n^3}{\cancel{7^n}} = \lim_{n\to\infty} \frac{7n^3}{(n+1)^3} = 7$$

$n^3 + 3n^2 + 3n + 1$

No values of K