Math 163 - Calculus - Exam 2 - Guide **Name:** ______________________

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1. In each of the following, determine convergence/divergence. Indicate which test(s) you are using.

(1.1) Indicate absolute convergence, conditional convergence or divergence

$$\sum_{k=1}^{\infty} \frac{(-1)^n}{2n}$$ converges by A.S.T.

Alternating Series Test: it's alternating: $(-1)^n$

$\{\frac{1}{2n}\}$ decreasing sequence: $\approx \frac{1}{n}$

is it absolute or conditional

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n} \right| = \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

Harmonic Diverges

Integral Test: $\int_1^{\infty} \frac{1}{n} = \ln|n| \Big|_1^{\infty} = \infty$

(1.2) $$\sum_{k=2}^{\infty} \frac{2k\sqrt[3]{k}}{3k^2+5k+1}$$

Comparable: $\frac{2k\sqrt[3]{k}}{3k^2} = \frac{2 \cdot k^{4/3}}{3\ k^2} = \frac{2}{3k^{2/3}}$ ---> div. by p-test, $p<1$

L.C.T.

$$\lim_{k\to\infty} \frac{\frac{2k\sqrt[3]{k}}{3k^2+5k+1}}{\left(\frac{2}{3k^{2/3}}\right)} = \lim \frac{\cancel{2}k\sqrt[3]{k}}{3k^2+5k+1} \cdot \frac{3k^{2/3}}{\cancel{2}} = \lim_{k\to\infty} \frac{3k^2}{3k^2+5k+1} \stackrel{L'H}{=} \frac{\infty}{\infty}$$

$$= \frac{6k}{6k+5} = \frac{\infty}{\infty}$$

$\stackrel{L'H}{=} 1 > 0 \Rightarrow$ same behavior series diverges by L.C.T.

(1.3) $$\sum_{k=1}^{\infty} \cos\left(\frac{1}{k^2}\right)$$

think: divergence test

$$\lim_{k\to\infty} \left\{\cos\left(\frac{1}{k^2}\right)\right\} = \left\{\cos\left(\frac{1}{\infty}\right)\right\} = \cos(0) = 1$$

— diverges by div. test —

note:

$\sum \sin\left(\frac{1}{k^2}\right)$

$\{\sin \frac{1}{k^2}\} \longrightarrow \sin(0) = 0$

div. test doesn't apply.

(1.4) $\sum_{k=1}^{\infty}\left[\frac{8}{5}-\frac{\sqrt[k]{5}}{2}\right]^k$

$\frac{1}{\infty} = 0$

Root test:

$$\lim_{k\to\infty}\left[\left(\frac{8}{5}-\frac{\sqrt[k]{5}}{2}\right)^k\right]^{1/k} = \lim_{k\to\infty} \frac{8}{5} - \frac{\sqrt[k]{5}}{2} = \frac{8}{5} - \frac{5^{\lim_{k\to\infty}\frac{1}{k}}}{2} = \frac{8}{5}-\frac{1}{2}$$

$\sqrt[k]{5} = 5^{1/k}$

$$= \frac{16-5}{10} = \frac{11}{10}$$

$\frac{11}{10} > 1 \Rightarrow$ diverges

just like r in geom. series or ratio test

(1.5) $\sum_{k=2}^{\infty}\frac{7k}{k^3+17}$

comparable: $\frac{7k}{k^3} = \frac{7}{k^2} \rightsquigarrow \sum \frac{7}{k^2}$ conv. by p-test

$$\frac{7k}{k^3+17} < \frac{7k}{k^3} = \frac{7}{k^2} \rightsquigarrow \text{conv.}$$

By Direct Comp. Test $\Rightarrow$ our series converges

(1.6) $\sum_{k=0}^{\infty}\frac{5^{3k}}{(2k)!}$

ratio

$2(k+1)! = (2k+2)!$
$= (2k+2)(2k+1)(2k)!\ldots$

$$\lim_{k\to\infty}\frac{\frac{5^{3(k+1)}}{(2(k+1))!}}{\frac{5^{3k}}{(2k)!}} = \frac{5^{3k+3}}{(2k+2)(2k+1)\cancel{(2k)!}}\cdot\frac{\cancel{(2k)!}}{\cancel{5^{3k}}} = \lim \frac{5^3}{(2k+2)(2k+1)} = 0$$

$5^{3k+3} = \cancel{5^{3k}}\cdot 5^3$

$\Rightarrow$ conv.

(1.7) $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$

Comparable: $\frac{1}{k}$
use argument

L.C.T.

$$\lim_{k \to \infty} \frac{\sin(\frac{1}{k})}{\frac{1}{k}} = \frac{0}{0}$$

(L'H)

$$\frac{\cos(\frac{1}{k}) \cdot \cancel{(-\frac{1}{k^2})}}{\cancel{(\frac{-1}{k^2})}} = \lim_{k \to \infty} \cos\left(\frac{1}{k}\right) = 1$$

chain rule

$1 > 0 \Rightarrow$ same behavior. Since $\sum \frac{1}{k}$ div. our series div. by L.C.T test

2. Prove the following statement: If $\underbrace{\sum a_n \text{ converges}}_{\text{assume}}$, then $\overline{\lim_{n\to+\infty} a_n = 0}^{\text{show}}$

ex $S_k = \frac{1}{k} + \frac{5}{k^2}$

Assume $\sum a_n$ converges
then $\lim_{k \to \infty} S_k = L$ where S_k = partial sum.

ex $S_{k-1} = \frac{1}{k} + \frac{5}{(k-1)^2}$

thus, $\lim_{k \to \infty} S_{k-1} = L$

note: $a_k = S_k - S_{k-1}$

ex
$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$

$$\lim_{n \to \infty} a_n = \lim (S_n - S_{n-1})$$
$$= \underbrace{\lim S_n}_{= L} - \underbrace{\lim S_{n-1}}_{= L} = 0$$

3. Find the value of the convergent series below:

(3.1) $\displaystyle\sum_{k=1}^{+\infty} \frac{2^{k+1}}{3^{k-1}}$

(3.2) $\displaystyle\sum_{k=2}^{+\infty} \left[64^{1/k} - 64^{1/(k+2)}\right]$

$S_2 = 64^{1/2} - 64^{1/4}$

$S_3 = 64^{1/2} - 64^{1/4} + 64^{1/3} - 64^{1/5}$

$S_4 = 64^{1/2} - 64^{1/4} + 64^{1/3} - 64^{1/5} + 64^{1/4} - 64^{1/6}$

$\uparrow$ dies later

dies 2 steps late

$S_5 = \ldots\ldots + 64^{1/5}$

$S_k = 64^{1/2} + 64^{1/3} - 64^{1/k+1} - 64^{1/k+2}$

$\lim\limits_{k\to\infty} 64^{1/2} + 64^{1/3} - \underset{1}{\underset{\shortparallel}{64^0}} - \underset{1}{\underset{\shortparallel}{64^0}} = 8 + 4 - 1 - 1 = 10$