MA163

Taylor / MacLaurin Series
- Homework (Achieve) dates will change.

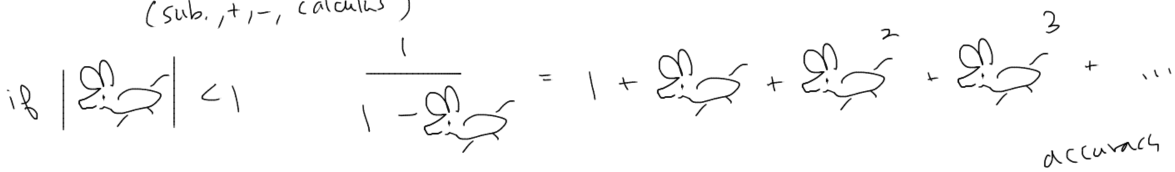
week 8
" 9
" 10 (Taylor) Exam 3
11
12
13 (Apps)
14 Exam 4

Final

warm-up

we know: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$ ⊛ when $|x|<1$

and "manipulations are valid" on the interval of convergence
(sub., +, −, "calculus")



accuracy improves $\longrightarrow$

so sub $x = -x^2$ above ⊛

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

$\downarrow$ integrate wrt. x

$$\tan^{-1}x = \int \frac{1}{1+x^2}\,dx = \int 1 - x^2 + x^4 - x^6 + x^8 - \dots\, dx$$

since the series converges when $|x|<1$, we can integrate piece by piece

$$= \int 1\,dx - \int x^2\,dx + \int x^4\,dx + \dots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

so sub $x=1$ into both
$\downarrow$

$$\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

$\downarrow$ mult. by 4

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \dots$$

Motivation for Taylor Series (and MacLaurin)

Most functions can be represented as power series

$x^1, x^2, x^3, \ldots$

most complicated things can be viewed as a predictable combination of simple things

If $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$

$f(0) = a_0$ ← only thing not killed by $x = 0$

$7x \to 7$

$a_1 x \to a_1$

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots$

$f'(0) = a_1$

$f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \ldots$

$f''(0) = 2a_2$

$f'''(x) = 6a_3 + 24a_4 x + 60a_5 x^2 + \ldots$

$f'''(0) = 6a_3$

combine what we've learned

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \ldots$

$0!$ $1!$ $2!$ $3!$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ MacLaurin series

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} \cdot (x-c)^n$ Taylor Series

Ex: Write out the first four terms of the MacLaurin Series of f(x) if:

f^{0}

$f(0) = 7$, $f'(0) = 7$, $f''(0) = 18$, $f'''(0) = 18$

$$f(x) = \sum_{n=0}^{\cancel{\infty}\ 4} \frac{f^{(n)}(0)}{n!} x^n = \frac{f^{(0)}(0)}{\underbrace{0!}_{1=}} \overbrace{x^0}^{=1} + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3$$

$n=0$ $n=1$

$$= 7 + 7x + \frac{18}{\underset{=2}{2}} x^2 + \frac{18}{\underset{=6}{3!}} x^3$$

$$f(x) \approx 7 + 7x + 9x^2 + 3x^3$$