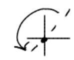
Mon. wk 9

MacLaurin Series : $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ Taylor Series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

Consider : $\sin(x)$ & $\cos(x)$

we know :

$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\sin(0) = 0$	$\cos(x)$ is even
$\frac{d}{dx}(\cos(x)) = -\sin(x)$	$\cos(0) = 1$	$\sin(x)$ is odd



write out MacLaurin Series ($a = 0$) for $\sin(x)$:

sub; $x = a = 0$

$f(x) = \sin(x)$ ⓪ $f(0) = \sin(0) = 0$

$f'(x) = \cos(x)$ ① $f'(0) = 1$

$f''(x) = -\sin(x)$ ② $f''(0) = 0$

$f'''(x) = -\cos(x)$ ③ $f'''(0) = -1$

$f^{(IV)}(x) = \sin(x)$ ④ $f^{(IV)}(0) = 0$

start over

$$f(x) = \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{IV}(0)x^4}{4!}$$

$$= 0 + x + \frac{(-1)x^3}{3!} + 0 \dots$$

$$f^{(n)}(0) = \begin{cases} 0 & n = \text{even} \\ (-1)^{n-1} & n = \text{odd} \end{cases} = \quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

So ... $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ (From Ratio Test we see this converges $\forall x$) $\Rightarrow$ manipulation are valid

$\downarrow d/dx$

$$\boxed{\cos(x) = 1 - \frac{\cancel{3}x^2}{3! = \cancel{3}\cdot 2!} + \frac{5x^4}{5! \dots 5\cdot 4!} - \frac{7x^6}{7! \dots 7\cdot 6!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}}$$

this is the MacLaurin series for $\cos(x)$

$f(x) = \cos x$ $f(0) = 1$

$f'(x) = -\sin x$ $f'(0) = 0$

$f''(x) = -\cos x$ $f''(0) = -1$

$f'''(x) = \sin x$ $f'''(0) = 0$

$f^{(IV)}(x) = \cos x$ $f^{IV}(0) = 1$

$$\cos(x) = 1 + 0\cdot x + \frac{(-1)}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$ same

take d/dx

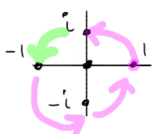
$$-\sin x = 0 - \frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \dots$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!}$$

3 basic Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin x = \sum_{n=0} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0} (-1)^n \frac{x^{2n}}{(2n)!}$$

In physics / engineering complex #'s are used to express rotations.



$\mathbb{C}$

$\bullet (x,y) = x + iy$

$$i \cdot i = i^2$$

$$-i(i) = -(i^2) = -(-1) = 1$$

$$= -(i \cdot i) = -1 \, i^2$$

$0i = 0$

$(1,0) = 1$

$(0,1) = 0x + i(1) = i$

$(-1,0) = -1$

$(0,-1) = 0x + i(-1) = -i$

$$1 \xrightarrow[\text{by } i]{\text{mult.}} i \xrightarrow[\text{by } i]{\text{mult.}} -1 \xrightarrow[\text{by } i]{\text{mult.}} -i \xrightarrow[\text{by } i]{\text{mult.}} 1$$

$$i^0 \qquad i \qquad i^2 \qquad i^3 \qquad i^4$$

Sub. $i\theta = x$ in e^x series above.

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \underbrace{\frac{(i\theta)^0}{0!}}_{n=0} + \underbrace{\frac{(i\theta)^1}{1!}}_{n=1} + \underbrace{\frac{(i\theta)^2}{2!}}_{n=2} + \frac{(i\theta)^3}{3!}$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots$$

reorder terms based on odd/even exponents

$$= i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + \ldots + 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!}$$

expand i^n: $i^2 = -1$, $i^6 = -1$, $i^{10} = -1$ | $i = i$, $i^5 = i$

$i^4 = 1$, $i^8 = 1$, $i^{12} = 1$ | $i^3 = -i$, $i^7 = -i$

$$= \underbrace{i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \ldots}_{} + \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots}_{}$$

$$\boxed{e^{i\theta} = i\sin(\theta) + \cos\theta}$$ Euler's Formula

sub $\theta = \pi$, to get wildest eq'n in all of math.

$$e^{i\pi} = i\sin\pi + \cos\pi = i0 + (-1) \Rightarrow e^{i\pi} = -1$$

one equation, five of the most important numbers

$$\boxed{e^{i\pi} + 1 = 0}$$