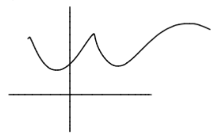
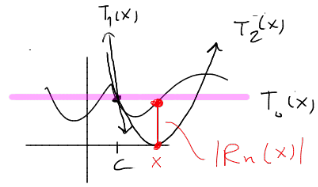
thurs wk 9

Taylor's Remainder theorem & Making Estimates

Idea: given $f(x)$





approximate the graph starting @ $x=c$

the more "n" the better the approx.

Taylor's Remainder Thm: Gives a bound on "n":

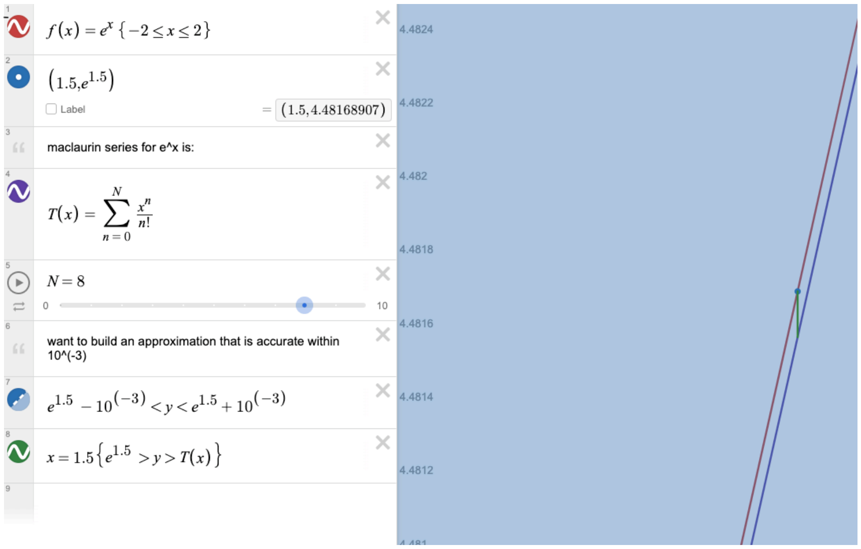
$$|R_n(x)| = |f(x) - T_n(x)| \le \frac{f^{(n+1)}(u)\,|x-c|^{n+1}}{(n+1)!} = \frac{M\,|x-c|^{n+1}}{(n+1)!}$$

(where u is the x-value that maximizes the $(n+1)$-derivative)

Error in Approximating $f(x)$ with $T_n(x)$

where $M \ge f^{(n+1)}(x)$ whenever x is close to c

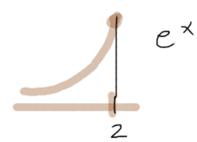
$|x-c| < d$



ex) let $f(x) = e^x$ on $[-2,2]$. Approximate $e^{1.5}$ w/ error less than 10^{-3}.

1. decide what c is. Since 1.5 is close-ish to 0, (MacLaurin Series are nice), let's use $c=0$.
2. Find MacLaurin Series for e^x

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$



3. Taylor's Rem. Theorem $c=0$ $\quad |R_n(x)| \le \dfrac{M x^{n+1}}{(n+1)!}$. What is M?

$M = \# \ge (n+1)$-derivative evaluated in the region $= e^x$ where $x \in [-2,2]$.

what # is $\ge e^x$ if $x \in [-2,2]$

$e^x < 3^x$, Choose $M = 3^2 = 9$

4. want to approx $e^{1.5}$, $1.5 = x$

$$\underbrace{|R_n(1.5)|}_{\text{error}} \le \frac{9(1.5)^{n+1}}{(n+1)!} \overset{\text{want}}{<} 10^{-3}$$

5. solve the inequality $\quad \dfrac{(1.5)^{n+1}}{(n+1)!} < \dfrac{10^{-3}}{9} = \dfrac{1}{9000}$

6. $\underset{1.5^n \cdot 1.5}{\overset{\shortparallel}{1.5^{(n+1)}}} < \dfrac{(n+1)!}{9000} \Rightarrow 1.5^n < \dfrac{(n+1)!}{\underset{1.5 \cdot 9 \cdot 1000}{\overset{\shortparallel}{13500}}}$

7. $n = 8$ works!

8. $e^{1.5} \approx \sum_{n=0}^{8} \frac{(1.5)^n}{n!} = 4.185$ w/in 10^{-3} of $e^{1.5}$.

Note: different choice of M, $M = 100$.

$$|R_n(1.5)| \le \frac{100(1.5)^{n+1}}{(n+1)!} \rightsquigarrow 1.5^n < \frac{(n+1)!}{1.5 \cdot 100 \cdot 1000}$$

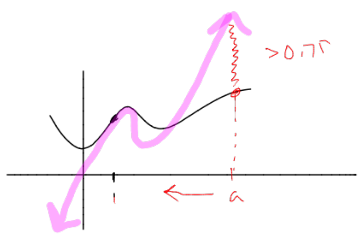
· Taylor's Rem Theorem is used to approx functions @ specific values.
eg, $e^{1.5}$.

· To find the maximum x s.t. $|R_n(x)| < 10^{-3}$ you can use a graph of $R_n(x)$.

(ex) Compute T_3 for $f(x) = 3\sqrt{x}$ centered @ $c = 1$.

... turns out this = $T_3(x) = 3 + \frac{3}{2}(x-1) - \frac{3}{8}(x-1)^2 + \frac{3}{16}(x-1)^3$

Find the largest a s.t. on $[1, a]$ the error ≤ 0.75



$R_n(x) = 3\sqrt{x} - 3 + \frac{3}{2}(x-1) - \frac{3}{8}(x-1)^2 + \frac{3}{16}(x-1)^3$

↑ height = error incurred @ x.