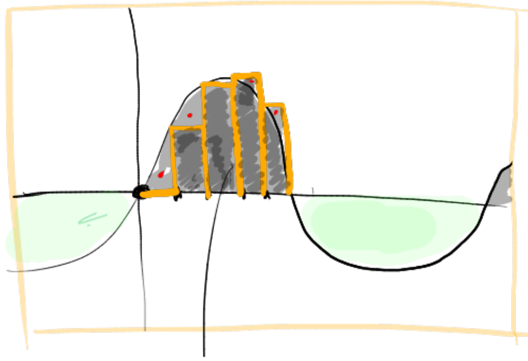
Applications of Integrals –

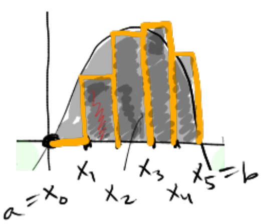
1. area

For each region divide into equal intervals



Each interval determines a rectangle, whose height is determined by a choice (left/right/midpoint) & the function.

The total area of the rectangles is an approximation to the area of this region.



$$A \approx \underbrace{0 \cdot \Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x}$$

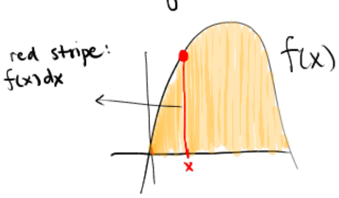
$$A \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

width $= \Delta x = \frac{b-a}{n}$

$(n=5)$

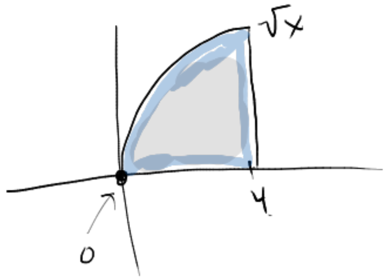
The precise area is $A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$

why?



1. What is $f(x)$?
   A: Height of graph above x.
2. What is $f(x_i)\Delta x$?
   A: Area of small rectangle above x_i
3. What is $f(x)dx$?
   A: Area of infinitesimal rectangle above x.
4. What is $\int_a^b f(x)dx$? A: Infinite sum of infinitesimal areas, from a to b.

Ex



what's the area? $\sqrt{x} = x^{1/2}$

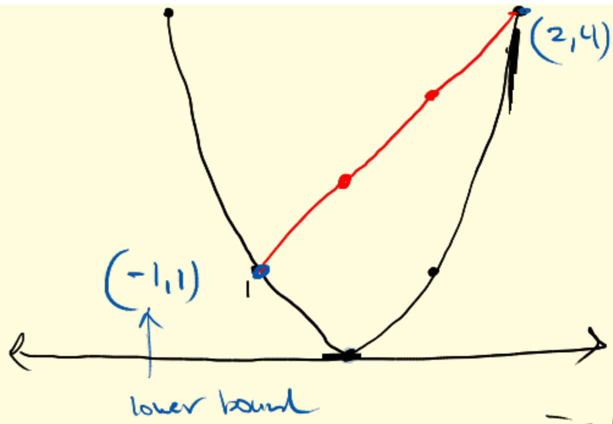
$$\int_0^4 \sqrt{x}\, dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^4 = \frac{2}{3}(4)^{\frac{3}{2}} - \frac{2}{3}0^{\frac{3}{2}}$$

$$= \frac{2}{3}\sqrt{4^3}$$

$$= \frac{2}{3}\sqrt{64} = \frac{2}{3}\cdot 8$$

$$= \frac{16}{3}$$

$$= 5\frac{1}{3}$$



$$\underbrace{\int_{-1}^{2} x+2\,dx}_{\text{area under red}} - \underbrace{\int_{-1}^{2} x^2\,dx}_{\text{area under black}}$$

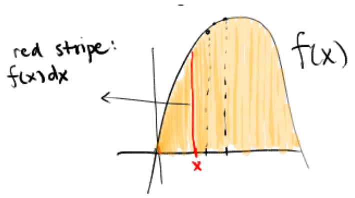
$$= \int_{-1}^{2} x+2-x^2 \; dx$$

$$= \frac{x^2}{2}+2x-\frac{x^3}{3}\Big|_{-1}^{2} = \frac{4}{2}+4-\frac{8}{3} - \underbrace{\left(\frac{1}{2}+2(-1)-\frac{(-1)^3}{3}\right)}_{\text{plug in } -1}$$

$$= 6-\frac{8}{3} - \left(\frac{1}{2}-2+\frac{1}{3}\right)$$

$$= 6-\frac{8}{3} \quad +1.5-\frac{1}{3}$$

$$7.5 - 3 = 4.5$$

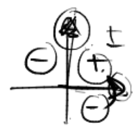


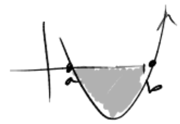
1. Rectangles eventually becomes so thin that left/right/midpoint distinction vanishes $\Rightarrow$ doesn't matter $\int_a^b f(x)\,dx$

2. The definite integral is an oriented quantity

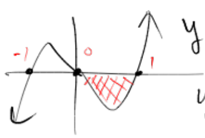
$$\int_a^b f(x)\,dx$$

(i)  $\longrightarrow$ def. integral is positive



(ii)  def. integral is negative

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Ex  $y = x(x^2-1)$

$y = x^3 - x$

evaluated at 1, subtract evaluated @ 0

plug in 1   plug in 0

$$\int_0^1 x^3 - x\,dx = \frac{x^4}{4} - \frac{x^2}{2}\Big|_0^1 = \left(\frac{1^4}{4} - \frac{1^2}{2}\right) - \left(\frac{0}{4} - \frac{0}{2}\right)$$

$$= \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \Rightarrow \text{area is below x-axis.}$$

$$\int_0^{2\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{2\pi} = -\cos(2\pi) - \left(-\cos(0)\right) = -1 - (-1) = 0$$

So to compute area under curve of $y = \sin(x)$ for $(0, 2\pi)$ do:

$$\int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$$

b/c this is negative

$$-\cos(x)\Big|_0^{\pi} + \cos(x)\Big|_{\pi}^{2\pi}$$

$$-(\cos(\pi) - \cos(0)) + \cos(2\pi) - \cos(\pi)$$

$$-(-1-1) + 1 - (-1)$$

$$-(-2) + 2 = 4$$

(iii) $$\int_a^b f(x)\,dx = -\int_b^a f(x)$$

(iv) Suppose $F'(x) = f(x)$. That is $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{(FTC)}$$

as $\Delta x \to 0$, the change in area approaches the height of the function

(v) $$g(x) = \int_a^x f(u)\,du \Rightarrow g'(x) = f(x)$$

$g(x)$ = area under $f(x)$ from a to x.

$$g(x^2) = \int_a^x f(u)\,du$$

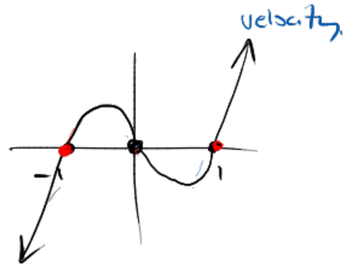
$$(g(x^2))' = f(x^2)\cdot 2x$$

Q: How is the area changing wrt x?

A: It's changing by the value $f(x)$

Q. What is the change in area for small Δx?

A: $f(x)\Delta x$. as $\Delta x \to 0$, Δx becomes dx so $f(x)\Delta x \to f(x)dx$



$x^3 - x = x(x^2-1)$

What is your displacement
(How far from home
are you at
$t=0$,
$t=1$
$t=2$?

start

$$\int_{-1}^{0} x^3 - x \, dx = \frac{x^4}{4} - \frac{x^2}{2}\Big|_{-1}^{0} = -\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4}$$

$$\int_{-1}^{1} x^3 - x = \frac{x^4}{4} - \frac{x^2}{2}\Big|_{-1}^{1} = \left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) = \varnothing \quad !!$$

$$\int_{-1}^{2} x^3 - x$$

$$\frac{x^4}{4} - \frac{x^2}{2}\Big|_{-1}^{2}$$

$$\frac{16}{4} - \frac{4}{2} - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$2 - \left(-\frac{1}{4}\right) = 2\frac{1}{4}$$