MAS41 - w/c - W

Ultimate Equality

$A \asymp B \iff \lim\limits_{\epsilon \to 0} \frac{A}{B} = 1$ $\qquad A = f(\epsilon),\ B = g(\epsilon)$

Prop. $\asymp$ is an equivalence relation

proof: R, S, T

Reflexivity: Show $A \asymp A$

$$\lim_{\epsilon \to 0} \frac{A}{A} = \lim_{\epsilon \to 0} 1 = 1 \iff A \asymp A$$

Symmetry: Show $A \asymp B \Rightarrow B \asymp A$

Assume $A \asymp B$

$\Rightarrow \lim\limits_{\epsilon \to 0} \frac{A}{B} = 1$

WTS $\lim\limits_{\epsilon \to 0} \frac{B}{A} = 1$

examine

$$\lim_{\epsilon \to 0} \frac{B}{A} = \lim_{\epsilon \to 0} \frac{B}{A} \cdot 1 = \lim_{\epsilon \to 0} \frac{B}{A} \cdot \lim_{\epsilon \to 0} \frac{A}{B} = \lim \frac{B}{A} \cdot \frac{A}{B}$$

$$= \lim 1 = 1$$

Transitivity Exercise

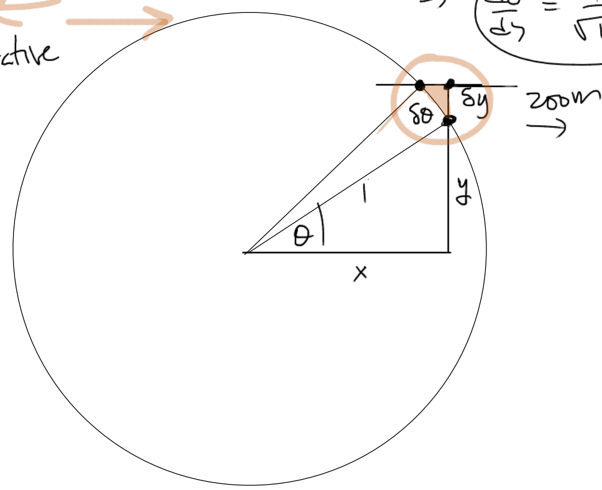
use $\approx$ to show $\frac{d}{dy}(\arcsin y) = \frac{1}{\sqrt{1-y^2}}$ $\theta = \arcsin(y)$

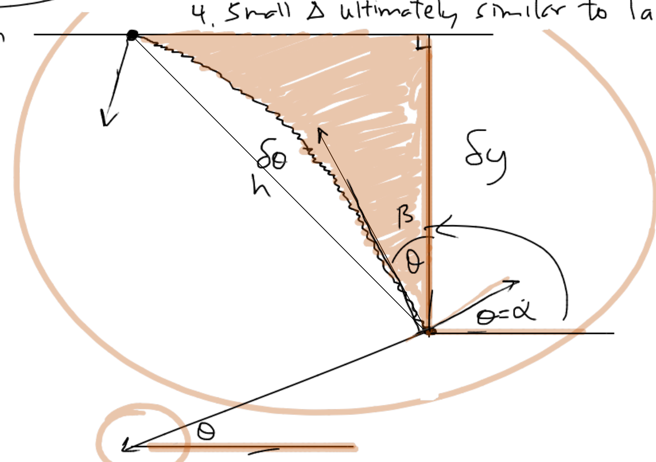
$\frac{d\theta}{dy} \approx \frac{\delta\theta}{\delta y} \overset{\text{similar}}{\approx} \frac{1}{x} \underset{P.T}{=} \frac{1}{\sqrt{1-y^2}}$

derivative

$\Rightarrow$ $\frac{d\theta}{dy} = \frac{1}{\sqrt{1-y^2}}$

1. let y vary
2. As $\delta\theta \to 0$. $h \to$ tangent line to circle
3. $\frac{\pi}{2}$ rotation takes $\alpha \mapsto \beta$.
   b/c tangent $\perp$ radius $\Rightarrow$ $\beta = \theta$
4. Small Δ ultimately similar to large.





Text 1.4

Focus: curves (1-dimensions) | Setting: these curves & surfaces
surfaces (2-dimensions) | often sit wildly in 3-dim's.

There's a remarkable interplay b/w the geometry of the curves & surfaces themselves
(distances within surface, area, length along curve)
and
how they sit inside ambient 3-dim'l space.

Recall Harriot's Theorem: $\varepsilon(\Delta) = \frac{1}{R^2} A(\Delta)$

Lambert extends this Hyperbolic space

$$\boxed{\varepsilon(\Delta) = K A(\Delta)}$$

constant

$K > 0 \Rightarrow$ sphere

$K < 0 \Rightarrow$ hyperbolic

$K = 0 \Rightarrow$ Euclidean

- Since angle sum ≥ 0 (If $\varepsilon(\Delta) < 0$)

$\varepsilon(\Delta) \geq -\pi \Rightarrow$ Hyperbolic $\Rightarrow K$ const < 0

Lambert, angles sum > 0

$\varepsilon(\Delta) = K \cdot A(\Delta) \geq -\pi$

OFTEN we use models of $\mathbb{H}^2$ that have $K = -1$

$\div$ by $K \Rightarrow \boxed{A(\Delta) \leq \frac{\pi}{|K|}}$

$\Rightarrow$ No large Δ's in $\mathbb{H}^2$

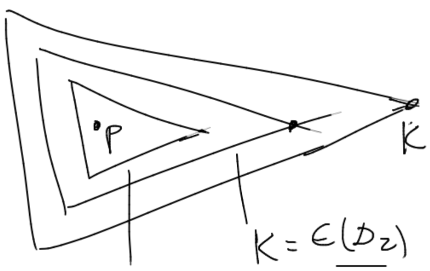
Ch. 2 Gaussian Curvature

Recall: $\varepsilon(\Delta) = \frac{1}{R^2} A(\Delta)$ Harriot

$\varepsilon(\Delta) = K A(\Delta)$ Lambert

both involve a constant, that depends on a triangle

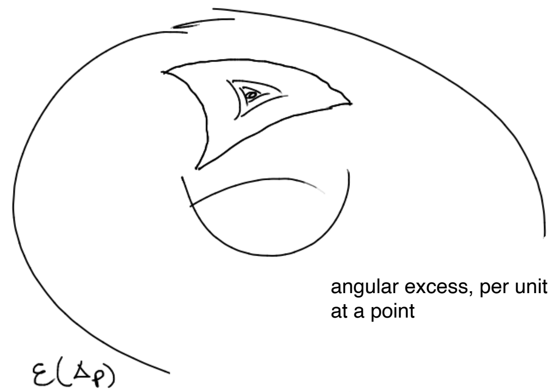
Gauss sought a constant depending only a point to describe how a surface curved.



$K = \frac{\varepsilon(\Delta_1)}{A(\Delta_1)}$

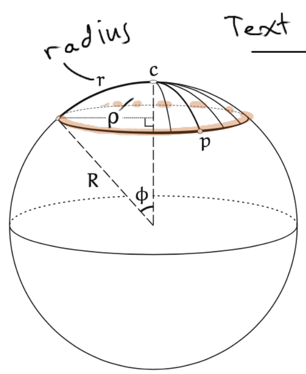
$K = \frac{\varepsilon(\Delta_2)}{A(\Delta_2)}$

$K = \frac{\varepsilon(\Delta_3)}{A(\Delta_3)}$



angular excess, per unit area, at a point

$$K(p) = \lim_{\Delta p \to p} \frac{\varepsilon(\Delta_p)}{A(\Delta_p)}$$



[2.4] *A circle of radius r on a sphere of radius R has circumference C(r), given by* $C(r) = 2\pi R \sin(r/R)$.

<u>Text 2.2</u> It's trivial to see the circumference of a circle of radius r, on a sphere of radius R is $C(r) = 2\pi R \sin(r/R)$

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \ldots$$

For small circles, $\phi \to 0$

$$\sin\phi = \phi - \frac{\phi^3}{6} \Rightarrow \frac{\phi^3}{6} = \phi - \sin\phi$$

Let's examine the difference b/w this circumf & it's Euclidean cousin.

$$2\pi r - 2\pi R \sin(r/R) = 2\pi R \left[\frac{r}{R} - \sin(r/R)\right]$$

$$\simeq 2\pi R \left[\frac{\left(\frac{r}{R}\right)^3}{6}\right] = \frac{\pi r^3}{3R^2}$$

Why:

$$2\pi r - C(r) \simeq \frac{\pi r^3}{3R^2} = \overset{=K}{\frac{1}{R^2}} \cdot \frac{\pi r^3}{3} = K \cdot \frac{\pi r^3}{3}$$

$K =$ curvature of sphere :)

$\equiv$ let's solve for it.

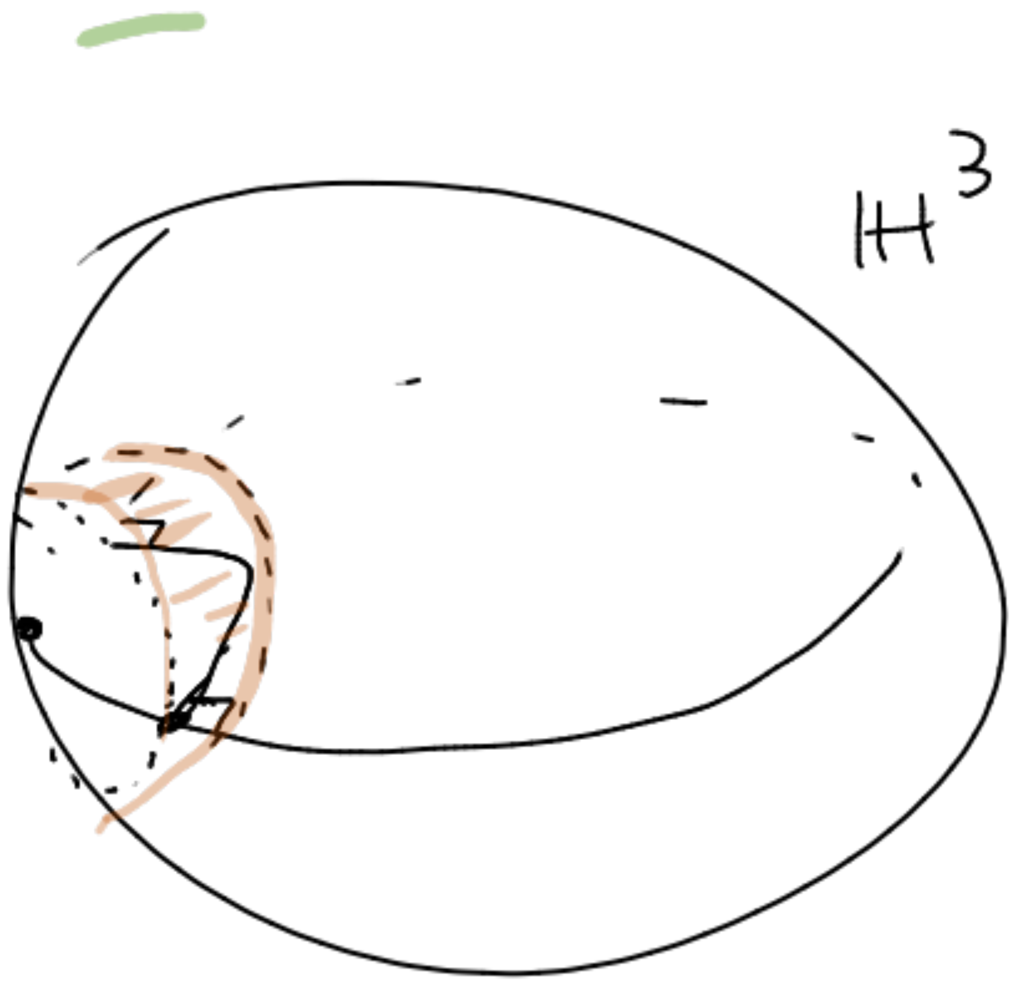
$$K \simeq \frac{3}{\pi r^3}\left[2\pi r - C(r)\right] = \frac{3}{\pi}\left[\frac{2\pi r - C(r)}{r^3}\right]$$

you can find how curved the space is, just by knowing the circumference of a circle

vice versa:

$$C(r) \simeq 2\pi r - K\frac{\pi r^3}{3}$$

you get circumf. from radius & curvature


$\mathbb{H}^3$

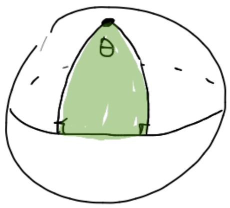
Local Gauss-Bonnet Theorem

For any Δ, any general surface

$$\varepsilon(\Delta) = \alpha + \beta + \gamma - \pi = \iint_\Delta K\,dA$$

angular excess of a triangle is exactly the total curvature contained inside it

Ex

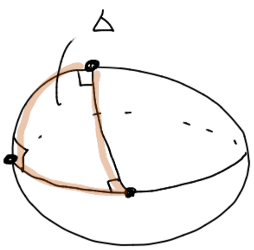


$$\varepsilon(\Delta) = \theta \overset{G.B.}{=} \iint_\Delta K\,dA = \iint_\Delta \frac{1}{R^2}\,dA = \frac{1}{R^2}\iint_\Delta dA$$

on the sphere what is $K = \frac{1}{R^2}$

$$= \frac{1}{R^2} A(\Delta)$$

Ex.



$$A(\Delta) = 4\pi R^2 \cdot \frac{1}{8} = \frac{\pi}{2} R^2$$

$$\varepsilon(\Delta) = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} - \pi = \frac{\pi}{2}$$

G.B. $\quad \frac{\pi}{2} = \iint_\Delta K\,dA = K\iint_\Delta dA = \frac{1}{R^2} A(\Delta) = \frac{1}{\cancel{R^2}} \cdot \frac{\pi}{2} \cdot \cancel{R^2}$