Curvature and Circulation

Definitions

- Let L be a small rectangle in the (u, v) plane, $\hat{L}$ its image on H^2.
- Let $V = (P, Q)$ be a vector field on the (u, v) plane induced by the metric on H^2:
  - $P = \frac{\partial_v(A)}{B}$
  - $Q = -\frac{\partial_u(B)}{A}$
- The **circulation** of V around the boundary of L is defined as:

$$C_V(L) = \oint_L V \cdot d\vec{s} = \oint_L P du + Q dv$$

Calculation

$$C_V(L) = \oint_L V \cdot dr = \oint_L \frac{\partial_v(A)}{B} du - \frac{\partial_u(B)}{A} dv$$

In the hyperbolic plane, $A = B = \frac{1}{v}$, so

$$C_V(L) = \oint_L v\partial_v \left(\frac{1}{v}\right) du - v\partial_u \left(\frac{1}{v}\right) dv$$

and this equals

$$C_V(L) = \oint_L -\frac{1}{v} du + 0 dv = \oint_L -\frac{1}{v} du$$

This reduces the circulation calculation to a line integral of a single-variable function around the rectangle L.

Let L have corners at (u, v), $(u + \delta u, v)$, $(u + \delta u, v + \delta v)$, and $(u, v + \delta v)$. Then these integrals become

$$\begin{aligned}
C_V(L) &= \int_u^{u+\delta u} -\frac{1}{v} du + \int_v^{v+\delta v} 0 dv + \int_{u+\delta u}^{u} -\frac{1}{v+\delta v} du + \int_{v+\delta v}^{v} 0 dv \\
&= -\frac{\delta u}{v} + \frac{\delta u}{v + \delta v} \\
&= \delta u \left(\frac{1}{v + \delta v} - \frac{1}{v} \right) \\
&= \delta u \left(\frac{v - (v + \delta v)}{v(v + \delta v)} \right) \\
&= -\frac{\delta u \delta v}{v(v + \delta v)}
\end{aligned}$$

Since the circulation per unit area is ultimately the curvature at a point, we divide by the area of $\hat{L}$, which is given by

$$\text{Area}(\hat{L}) = \iint_L \frac{1}{v^2} du dv = \frac{\delta u \delta v}{v^2}$$

So the curvature at the point (u, v) is given by

$$\begin{aligned}
K(u, v) &= \lim_{\text{Area}(\hat{L}) \to 0} \frac{C_V(L)}{\text{Area}(\hat{L})} \\
&= \lim_{\delta u, \delta v \to 0} \frac{-\frac{\delta u \delta v}{v(v+\delta v)}}{\frac{\delta u \delta v}{v^2}} \\
&= \lim_{\delta v \to 0} -\frac{v^2}{v(v + \delta v)} \\
&= \lim_{\delta v \to 0} -\frac{v}{v + \delta v} \\
&= -1
\end{aligned}$$