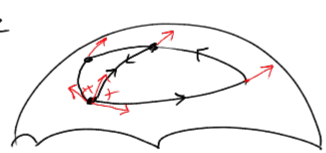
Wk 11 Mon

things we know:

Holonomy of loop:

① def: net change in tangent vector after circuit.

② property: holonomy is additive



$R(\Delta) = R(\Delta_1) + R(\Delta_2)$

③ Relation to curvature: $K(p) = \lim_{L_p \to p} \frac{R(L_p)}{A(L_p)}$ = holonomy per unit area

(From: $R(L) = K(L)$ (proved in 25?) + taking limits (i) do this $\frac{R(L)}{A(L)} = \frac{K(L)}{A(L)}$ (ii) then set $L_p = L$ and $L_p \to p$ $\lim_{L_p \to p} \frac{R(L_p)}{A(L_p)} = \lim_{L_p \to p} \frac{K(L_p)}{A(L_p)} = K(p)$

④ we also — know —

on a sphere $R(L) = K(L)$ curvature contained inside L

[true in gen'l but unproven]

(Ch. 25)

general geodesic triangle $R(\Delta) = \mathcal{E}(\Delta)$ angle defect

$\rightsquigarrow R(P_m) = \mathcal{E}(P_m)$

Harriot's thm. (on sphere) $\tilde{\mathcal{E}}(\tilde{P}_m) = \tilde{A}(\tilde{P}_m)$

$\Rightarrow$ on sphere: $\tilde{R}(\tilde{P}_m) = \tilde{A}(\tilde{P}_m)$

⑤ (i) S = surface, $\bar{n}$ unit normal field. S^2 sphere $n: S \to S^2$ spherical (Gauss) map

(ii) K_{ext} = extrinsic curvature ≡ local (signed) area expansion factor of Gauss map.

ex: S^2_{10} = , S^2_1 ⊘ — area: 4π $K_{ext} = \frac{1}{100} = \frac{1}{R^2}$

└ area: $4\pi(10)^2 = 400\pi$

(iii) ~ denotes a quantity on S^2

(iv) L = simple loop enclosing $\Omega \subset S$, $R(L)$ = holonomy ↺, $\tilde{R}(\tilde{L})$ holonomy on S^2.

(v) $K_{ext}(\Omega)$ total extrinsic curvature w/in Ω. $K_{ext}(\Omega) = \iint_\Omega K_{ext}\, dA$

$K(\Omega) = \iint_\Omega K\, dA = \underbrace{R(L)}_{\text{still unproven!}}$

what we really know

By appealing to def'n of K_{ext}

① $K_{ext}(\Omega) = \iint_\Omega K_{ext}\, dA = \iint_{\tilde{\Omega}} d\tilde{A} = \tilde{A}(\tilde{\Omega})$

total Amount of K_{ext} inside Ω

② $K_{ext} = \kappa_1 \cdot \kappa_2$ principal curvatures

ex  S^2_{10} $\kappa_1 = \frac{1}{R}, \kappa_2 = \frac{1}{R}$

{ principal curvatures ~ eigenvalues $\tilde{A} = \Pi \lambda_i$ }

① + ②

③ $\tilde{A}(\tilde{\Omega}) = \iint_\Omega \kappa_1 \kappa_2 \, dA$

④ Angle Defect on sphere $= \tilde{\mathcal{E}}(\tilde{P}_m) = \tilde{A}(\text{interior of } \tilde{P}_m)$ (generalized Harriot)

geodesic poly

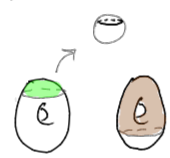
⑤ On a sphere: $\tilde{R}(\tilde{\Delta}) = \tilde{\mathcal{E}}(\tilde{\Delta}) \xrightarrow{\text{generalizes}} \tilde{R}(\tilde{P}_m) = \tilde{\mathcal{E}}(\tilde{P}_m)$

⑥ ④ + ⑤ $\Rightarrow$ $\tilde{R}(\tilde{P}_m) = \tilde{A}(\tilde{P}_m)$

⑦ when $\tilde{L}$ = non-geodesic, approximate w/ geodesic m-gon & let $m \to \infty$

Total amount of extrinsic curvature within Ω

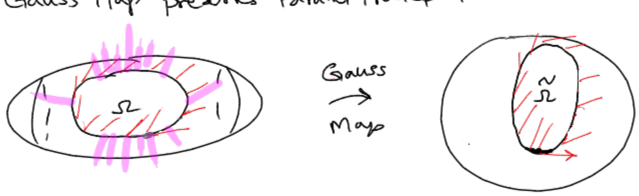
⑧ So ... $\iint_\Omega \kappa_1 \kappa_2 \, dA = K_{ext}(\Omega) = \tilde{A}(\tilde{\Omega}) = \tilde{R}(\tilde{L})$



the current theory avoids thinking of mixed signed curvature.

$L = \text{boundary}(\Omega)$

⑨ Gauss Map preserves Parallel Transport



\# w = vector v_0 parallel transported (tangent vector field)
Rate of change of $w_{\parallel}$ is always $\perp S$, i.e., the rate of change is along n b/c

Def'n $D_v(w) = 0$ (P.T. condition)
$= \nabla_v(w) - (n \cdot \nabla_v w)\, n$ (def'n cov. deriv)

★ the tangent planes @ p and $n(p)$ are parallel: ($\perp$ to normal vector)

By ★ and # the rate of change of $w_{\parallel}$ is $\perp$ to S^2 @ $n(p)$, thus $w_{\parallel}$ translated from $p \mapsto n(p)$ is parallel transported along $\tilde{L}$.

Beautiful theorem (1816): If a curved surface on which a figure is fixed takes different shapes in space then the surface area of the spherical map of the figure is always the same.

Theorem Egregium (1827): $\kappa_1\kappa_2 = \lim\limits_{\Delta p \to p} \dfrac{\mathcal{E}(\Delta p)}{A(\Delta p)} = K(p)$

Here's why:

① $R(L) = \tilde{R}(\tilde{L})$ Gauss Map preserves P.T., thus Holonomy

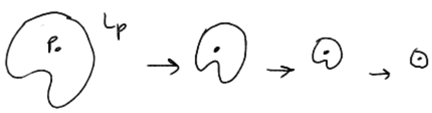
② $R(L) = \underbrace{\tilde{R}(\tilde{L})}_{\text{net rotation on sphere}} = \tilde{A}(\tilde{\Omega}) = K_{ext}(\Omega) = \underbrace{\iint_\Omega \kappa_1\kappa_2 \, dA}_{\text{total amt. of ext. curvature inside } \Omega \text{ back on } S.}$

$\downarrow$

defined intrinsically on S:
- tangent vector (topology)
- parallel transport (metric)

To P.T. approximate by geodesics! ~ metric } $R(L)$ depends only on the metric $\Rightarrow$ $R(L)$ isometry invariant $\Rightarrow$ Gauss's Beautiful Theorem of 1816

③ w/ L_p = small loop on S



While $\kappa_1 = \kappa_1(p)$, $\kappa_2 = \kappa_2(p)$ —

Using ② For very small L_p, κ_1, κ_2 are essentially constant

$R(L_p) \approx \iint_{\Omega} \underbrace{\kappa_1\kappa_2}_{\text{const.}} dA \approx \kappa_1\kappa_2 \iint_\Omega dA = \kappa_1\kappa_2 A(\Omega_p)$

$\Rightarrow \kappa_1\kappa_2 = \lim\limits_{L_p \to p} \dfrac{R(L_p)}{A(\Omega_p)}$ ← both intrinsic

↳ Finally proved

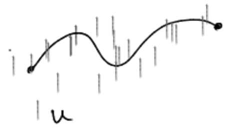
④ If L_p = small geodesic triangle $\Delta p \ni p$ since $R(\Delta_p) = \mathcal{E}(\Delta p)$ $\Rightarrow \lim\limits_{\Delta p \to p} \dfrac{\mathcal{E}(\Delta p)}{A(\Delta p)} = \kappa_1\kappa_2 = K(p)$ Theorema Egregium

$\Rightarrow \kappa_1\kappa_2$ is isometry invariant!

Holonomy proof of GGB

⓪ Hopf's Proof!

① Holonomy along open curve:



Introduce fiducial vector field
(no singularities @ curve)

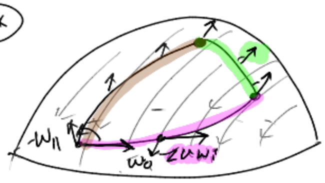
Index: $I_F(S)$ = # rotations of F rel. u.

of a singularity, vector field, surface

$I_F(S) = \frac{1}{2\pi} \delta_L(\angle uF)$

• surprising note: choice of vector field u doesn't matter

(ex)



$\Delta = K_1 + K_2 + K_3$

$R_u(K)$ = net change in angle $\angle uw_{\parallel}$ as we travel

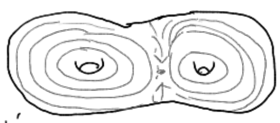
some open curve

{depends on our choice of u —
but independent of choice of $w_{\parallel}$}

$R(\Delta) = R_u(K_1) + R_u(K_2) + R_u(K_3)$ {sum is independent of u}

(ex) $S = \Sigma_g =$ 

F = vector field on it w/ finite # singularities



Goal: $K(S_g) \equiv \iint_{S_g} K\,dA = 2\pi \sum_i I_F(S)$

b/c LHS indep. of F $\Rightarrow$ sum is same $\forall$ vector fields