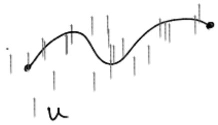
Ch. 2e Holonomy proof of GGB

⓪ Hopf's Proof!

① Holonomy along open curve:



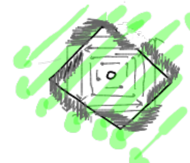
Introduce fiducial vector field
(no singularities)

• surprising note: choice of vector field u doesn't matter

Index of a singularity wrt vector field: $I_F(S)$ = # rotations of F rel. u.
(surface, vector field)

$$\boxed{I_F(S) = \frac{1}{2\pi}\delta_L(\angle uF)}$$

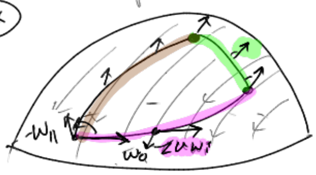
# of revolutions as we travel around L



$I_F(S) = 1$

Decompose Δ into 3 segments & introduce Fiducial vector Field.

(ex)



$\Delta = K_1 + K_2 + K_3$

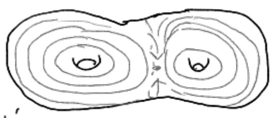
$$R_u(K) \equiv \delta_K(\angle u w_{11})$$

$R_u(K)$ = net change in angle $\angle u w_{11}$ as we travel (from u to w_{11})
(K = some open curve)

{depends on our choice of u — but independent of choice of w_{11}}

$R(\Delta) = R_u(K_1) + R_u(K_2) + R_u(K_3)$ {sum is independent of u}

(ex) $S = \Sigma_g =$ 



F = vector field on it w/ finite # singularities

b/c LHS indep. of F $\Rightarrow$ sum is same $\forall$ vector fields

Torus

$I_F(S) = 0$

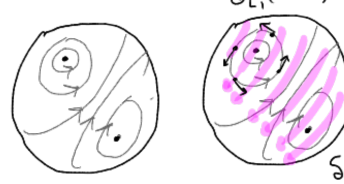
Has a non-vanishing vector field

$\chi(T) = 0 = \sum I_F(S)$

Hairy Billiard Ball theorem

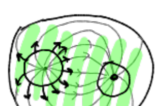
Any vector field on sphere must vanish somewhere.

$\delta_{L_1}(\angle uw) = 1$

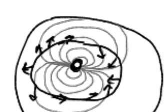


$I_F(S_1) = 1$
$I_F(S_2) = 2$

$\delta_{L_2}(\angle uw) = 1$



$I_F(S_1) = 1$ $I_F(S_2) = 1$



$I_F(S) = 2$

$\sum I_F(S) = 2$

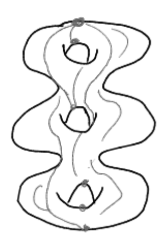
$$\boxed{\text{Goal:} \quad K(S_g) \equiv \iint_{S_g} K\,dA = 2\pi \sum_i I_F(S)}$$

LHS: indep. of vector field
so RHS must be too!

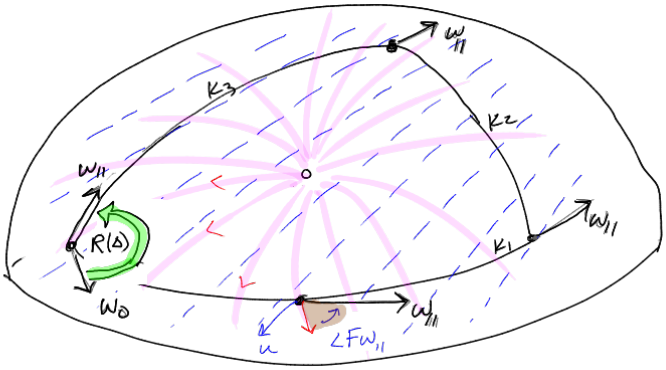
It's easy to build vector field on S_g s.t.

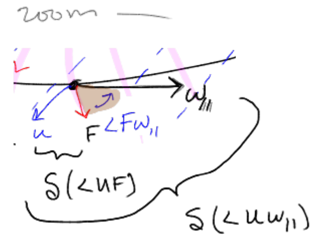
$\sum I_{F(S)} = 2 - 2g$. (Poincaré Hopf)

$= \chi(S_g)$ then: $K(S_g) = 2\pi\chi(S_g)$ G.G.B.



Honey Flow





Δ = geodesic triangle in S, w/ vector field F w/ singularity inside Δ

Fiducial Vector Field u:

$$K(\Delta) - 2\pi I_F(S) = R(\Delta) - 2\pi I_F(S)$$

curvature inside Δ

$$= R(\Delta) - \delta_\Delta(\angle uF)$$

revolutions ~~of~~ F relative to u

$$= \sum_j \left[R_u(K_j) - \delta_{K_j}(\angle uF)\right]$$

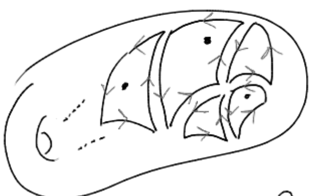
break into piece by piece

$$= \sum_j \left[\delta_{K_j}(\angle uw_{11}) - \delta_{K_j}(\angle uF)\right] = \underbrace{\sum \delta_{K_j}(\angle Fw_{11})}_{\Phi(K_j)\ \text{(notation)}}$$

net change in angle from u to w_{11}

rotation of w_{11} rel. to F - independent of u!

observe $\Phi(-K_j) = -\Phi(K_j)$



partition S into Polygons P_i containing @ most one singularity.

Sum over all polygons:

$$K(S_g) - 2\pi \sum I_F(S) = \sum_i K(P_i) - 2\pi \sum_i I_F(S)$$

$$= \sum_{\text{all polygons}} \Phi(K_j) = 0$$

b/c

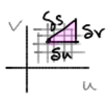
each edge K_j belongs to two adjacent P_i & is traversed in opposite directions

Geometric Proof of Metric Curvature Formula

$$K = -\frac{1}{AB}\left(\partial_v\left(\frac{\partial_v A}{B}\right) + \partial_u\left(\frac{\partial_v B}{A}\right)\right)$$

Construct Orthogonal Coord. System on S.

① On Surface, @ $\hat{p}$, draw non-intersecting curves near $\hat{p}$. "number" them (differentiably) and call them u-curves, defining the u-coord.



② Form orthogonal trajectories, similarly, giving v-coord

③ Let X measure dist. along v-curves on S. (horiz. lines in map)
Y u-curves vertical

④ Moving small dist. in map $\xrightarrow{\delta u}$ the corr. point moves a proportional distance on S.

$\delta X = A\delta u$ $\quad A$ = local horiz. scale factor

same for $\delta Y = B\delta v$

⑤ get $d\hat{s}^2 = A^2 du^2 + B^2 dv^2$ (Note: A, B both depend on u and v)

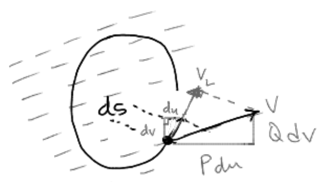
⑥ prepare to build a vector field in map plane

$$\begin{pmatrix} \dfrac{\partial_v A}{B} \\[2ex] \dfrac{\partial_u B}{A} \end{pmatrix}$$

Tool: the Circulation of a vector Field around a loop

Let $V = \begin{bmatrix} P(u,v) \\ Q(u,v) \end{bmatrix}$ be a vector field in (u,v) plane & $r = \begin{bmatrix} u \\ v \end{bmatrix}$ position vector of loop

Def'n: Circulation $C_L(V)$ of V around L is integral of component of V in the direction of L.

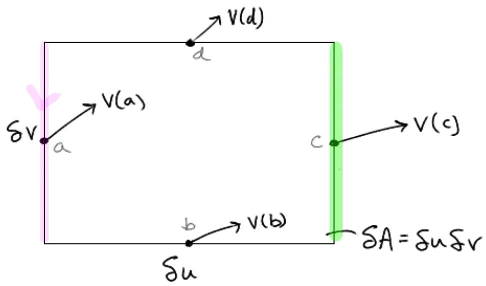


$ds = |dr|$

$dr = \begin{pmatrix} du \\ dv \end{pmatrix}$

$$C_L(V) = \oint_L V \cdot dr = \oint V_L \, ds = \oint_L P\,du + Q\,dv$$

For small rectangle R, calculate $C_R(V)$ (via Riemann Sum)



Contribution to $C_R(V)$ from $\uparrow$ is $\approx Q(c)\delta v$

from $\downarrow$ is $\approx -Q(a)\delta v$ ... total: $(Q(c) - Q(a))\delta v$

Same for horizontal: $(P(b) - P(a))\delta u$

$\Rightarrow C_R(V) \approx (Q(c) - Q(a))\delta v +$

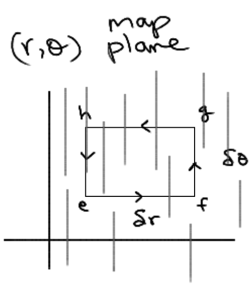
Holonomy In Flat Plane

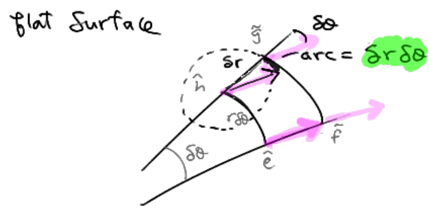
Recall: Holonomy Along Curve $= R_u(\hat{K}) \equiv \delta_{\hat{K}}(\angle u w_{\parallel})$ rotation of parallel t'd $w_{\parallel}$ rel. to u.

- when K = loop $R_u(\hat{L}) = R(\hat{L})$ becomes indep of u

(ex) S = flat plane w/ polar coords $u = r$, $v = \theta$, $A = 1$, $B = r$

$\leadsto \; d\hat{s}^2 = dr^2 + r^2 d\theta^2$





Rotation of $w_{\parallel}$ arises b/c $\hat{g}\hat{f}$ is longer than $\hat{h}\hat{e}$ — by

$r\delta\theta \longmapsto (r + \delta r)\delta\theta$

$= r\delta\theta + \underbrace{\delta r \delta\theta}_{\text{extra length}}$

so $\delta R_u(\hat{h}\hat{e}) = \delta\theta$

and $-\delta R_u(\hat{f}\hat{g}) \asymp \dfrac{\text{arc}}{\text{radius}} \asymp \dfrac{\text{increase in side length}}{\text{orthogonal side}} = \dfrac{\delta r\,\delta\theta}{\delta r} = \delta\theta$

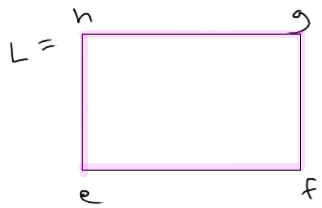
More generally, if $\delta Y \asymp B\delta v$ then $\delta^2 Y = \delta(\delta Y)$ $\left\{\begin{array}{l}\text{increase from}\\ u \mapsto u + \delta u\end{array}\right\}$

is $\delta Y \asymp [\partial_u B\, \delta v]\,\delta u$

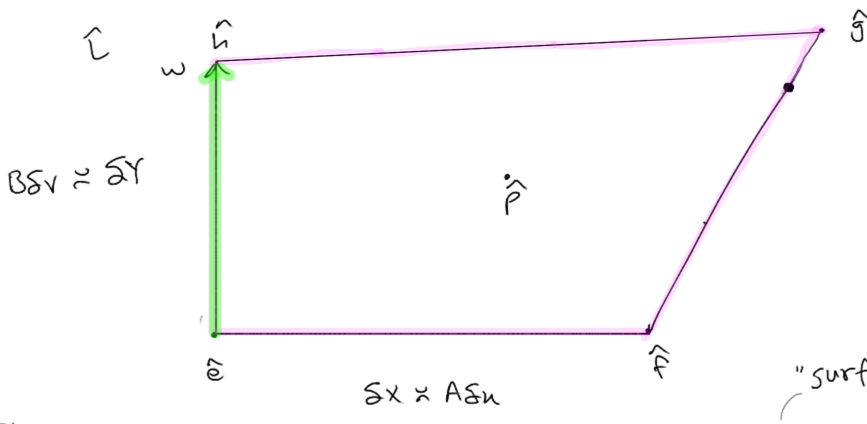
Note $\delta R_u(\text{edge}) = 0\,\delta r + (-1)\,\delta\theta = \begin{pmatrix}0\\-1\end{pmatrix}\cdot\begin{pmatrix}\delta r\\ \delta\theta\end{pmatrix}$ holonomy of loop = circulation of V around loop in map.

on map-plane (left)

Holonomy as Circulation of Metric Induced Vector Field in Map



(u,v)-map

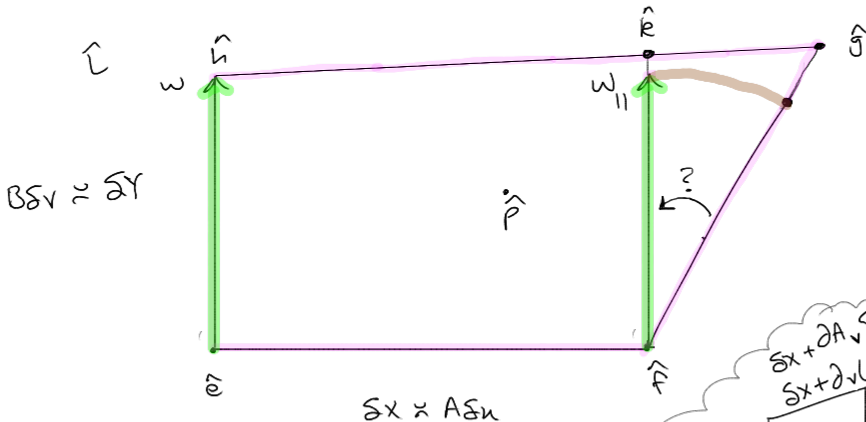


"surface"

actually since its curved, this is the local projection to tangent plane

Compute holonomy $R_{\hat{e}\hat{f}}(\hat{L})$

① Parallel Translate w to $\hat{f}$. (Recall, this is defined as: 'project to $T_{\hat{p}}(S)$ - translate Euclideanly - proj. back')



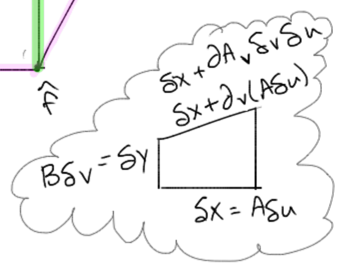
$? = \text{angle} = \dfrac{\text{arc length}}{\text{radius}}$

(since this is infintessimal ▱)

$\hat{h}\hat{k} \asymp \hat{e}\hat{f} \asymp \delta X$

$\hat{k}\hat{g} \asymp \delta(\delta X) = \delta^2 X$

$\asymp$ arc length



$\delta^2 X \asymp [\partial A_v \delta v]\delta u$

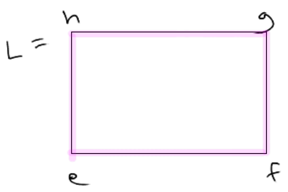
radius $\asymp \delta Y$

$$? = \frac{[\partial A_v \delta v]\delta u}{B\delta v} = \left(\frac{\partial A_v}{B}\right)\delta u$$

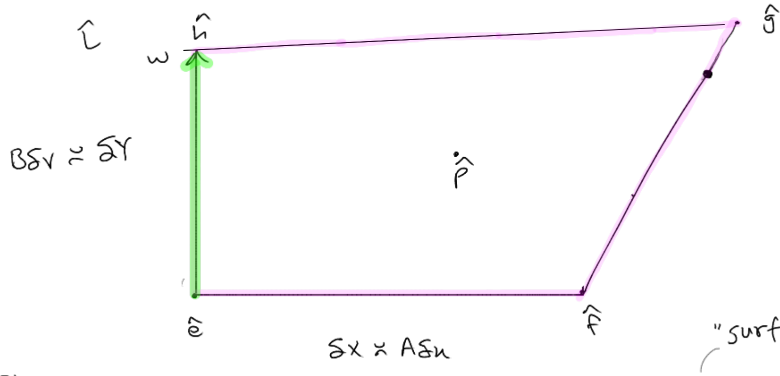
$R_{\hat{e}\hat{f}}(\hat{L})$ defined wrt fiducial vector field, which here is the (u,v)-curves

$$\boxed{R_{\hat{e}\hat{f}}(\hat{L}) = \left(\frac{\partial A_v}{B}\right)\delta u}$$

Similar for $R_{\hat e \hat h}(\tilde L)$



(u,v)-map

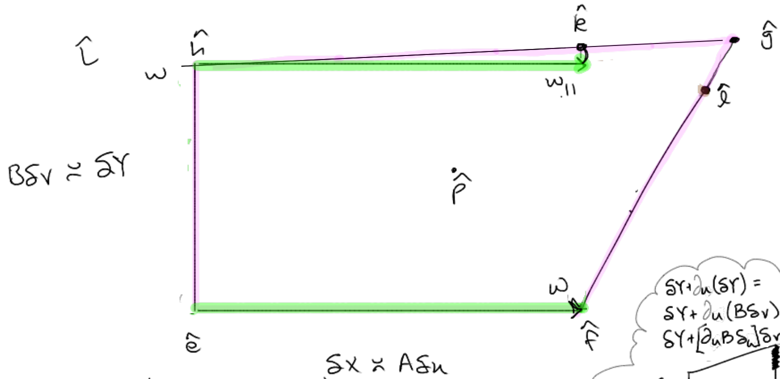


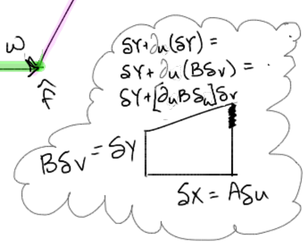
"surface"

actually since its curved, this is the local projection to tangent plane

Compute holonomy $R_{\hat e \hat h}(\hat L)$

① Parallel Translate w to $\hat h$. (Recall, this is defined as: 'project to $T_{\hat p}(S)$ - translate Euclideanly - proj. back')





(rotation wrt (u,v) curves)

$? = \text{angle} = \dfrac{\text{arc length}}{\text{radius}}$

(since this is infintessimal □)

$\hat f \hat \ell \approx \hat e \hat h \approx \delta Y$

$\hat \ell \hat g \approx \delta(\delta Y) = \delta^2 Y$

$\approx$ arc length

$\delta^2 Y \approx [\partial_u B\, \delta_u]\, \delta v$

radius $\approx \delta X$

negative (clockwise)

$$? = \frac{[\partial B_u \delta_u]\delta v}{A\delta u} = \left(\frac{\partial B_u}{A}\right)\delta v$$

$$\boxed{R_{\hat e \hat h}(\hat L) = -\left(\frac{\partial A_v}{B}\right)\delta u}$$

Putting Together: (linearity)

$$R(\hat{L}) = \left(\frac{\partial_v A}{B}\right)\delta u - \left(\frac{\partial_u B}{A}\right)\delta v = \begin{bmatrix} \frac{\partial_v A}{B} \\ -\frac{\partial_u B}{A} \end{bmatrix} \cdot \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}$$

V

to get holonomy of a general loop $\hat{L}$, integrate along its image in (u,v)-map.

$$\oint V \cdot \begin{bmatrix} \delta u \\ \delta v \end{bmatrix} = C_v(L)$$

$$R(\hat{L}) = C_v(L)$$

Pointwise: take limit $\hat{L} \to \hat{p}$

$$K(\hat{p}) = \lim_{\hat{L} \to \hat{p}} \frac{R(\hat{L})}{\delta \hat{A}} \asymp \frac{C_v(L)}{AB\,\delta A} = \frac{1}{AB}\left[\frac{C_v(L)}{\delta A}\right]$$

local circulation per unit area = curl of V

27.5 Geometric Proof of the Metric Curvature Formula

Now let us take the limit that the loop $\widehat{\mathrm{L}}$—with area $\delta\widehat{\mathcal{A}}$—shrinks down towards $\widehat{\mathrm{p}}$. The curvature $\mathcal{K}(\widehat{\mathrm{p}})$ is ultimately equal to the holonomy per unit area, so by combining (27.4) and (27.6) we obtain,

$$\mathcal{K}(\widehat{\mathrm{p}}) = \lim_{\widehat{\mathrm{L}} \to \widehat{\mathrm{p}}} \frac{\mathcal{R}(\widehat{\mathrm{L}})}{\delta\widehat{\mathcal{A}}} \asymp \frac{1}{\mathrm{AB}}\left[\frac{\mathcal{C}_{\mathrm{L}}(\mathbf{V})}{\delta\mathcal{A}}\right].$$

But (27.3) now tells us that the bracketed term on the right—namely, the local circulation per unit area—is none other than the *curl* of **V**.

This allows us to complete our geometric proof of the metric curvature formula, (27.1), one of the most elegant and *explicit* manifestations of Gauss's *Theorema Egregium* of 1827:

$$\begin{aligned}
\mathcal{K}(\widehat{\mathrm{p}}) &= \frac{1}{\mathrm{AB}}\left\{\text{curl of}\begin{bmatrix} (\partial_v \mathrm{A})/\mathrm{B} \\ -(\partial_u \mathrm{B})/\mathrm{A} \end{bmatrix}\right\} \\
&= \frac{1}{\mathrm{AB}}\left\{\partial_u\left[-\frac{\partial_u \mathrm{B}}{\mathrm{A}}\right] - \partial_v\left[\frac{\partial_v \mathrm{A}}{\mathrm{B}}\right]\right\} \\
&= -\frac{1}{\mathrm{AB}}\left(\partial_v\left[\frac{\partial_v \mathrm{A}}{\mathrm{B}}\right] + \partial_u\left[\frac{\partial_u \mathrm{B}}{\mathrm{A}}\right]\right).
\end{aligned}$$

$$K(\hat{p}) = \frac{1}{AB}\left\{\text{curl of}\begin{bmatrix} \partial_v A / B \\ -\partial_u B / A \end{bmatrix}\right\}$$

$$= \frac{1}{AB}\left\{\partial_u\left[\frac{-\partial_u B}{A}\right] - \partial_v\left[\frac{\partial_v A}{B}\right]\right\}$$

$$= -\frac{1}{AB}\left(\partial_v\left[\frac{\partial_v A}{B}\right] + \partial_u\left[\frac{\partial_u B}{A}\right]\right)$$