Curvature via Curl

ma541 - f25 - assignment 8

Find the curvature of the standard model spaces via their curl as follows.

1. Beginning with the metric for the sphere of radius (R):
$$ds^2 = R^2(\sin^2\phi\, d\theta^2 + d\phi^2)$$
find the metric-induced vector field $V = (P, Q)$ induced by the metric on S^2 where
$$V = \left(\frac{\partial_v(A)}{B}, -\frac{\partial_u(B)}{A}\right).$$
Let L be a small rectangle in the (u, v) plane, $\hat{L}$ its image on S^2. Use the circulation of V around the boundary of L to find an expression for the curvature at a point (θ, ϕ) on the sphere.

2. Repeat the process for the metric of the torus of revolution with parameters R and r:
$$ds^2 = (R + r\cos v)^2 du^2 + r^2 dv^2$$
to find an expression for the curvature at a point (u, v) on the torus.

   i) What is the sign of the curvature at points where $v = 0$ or $v = \pi$?

   ii) What is the sign of the curvature at points where $u = 0$ or $u = \pi/2$?

   iii) Explain how these signs relate to the orientation of the Gauss map at these points.

3. Repeat the process for the hyperbolic plane with metric:
$$ds^2 = \frac{1}{v^2}(du^2 + dv^2)$$
to verify that the curvature is -1 at all points.

4. (Optional) Use R/Python/MATLAB to visualize the vector fields on the sphere, torus & hyperbolic plane. Numerically approximate the curvature at various points. Compare your numerical results to your analytical expressions from parts 1-3.