Metric Curvature Formula.

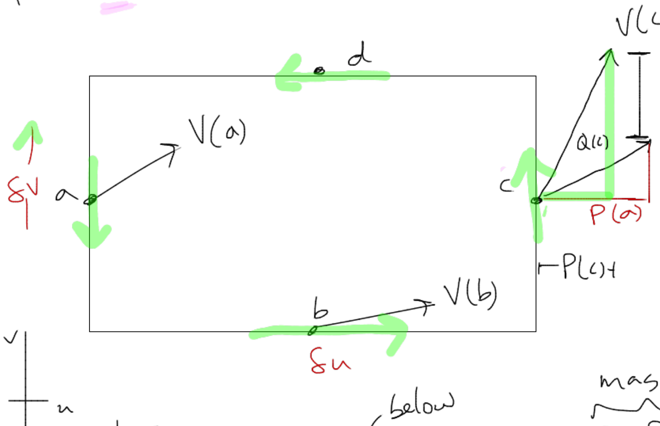
$$K = -\frac{1}{AB}\left(\partial_v\left(\frac{\partial_v A}{B}\right) + \partial_u\left(\frac{\partial_u B}{A}\right)\right)$$

$V = \begin{bmatrix} P(u,v) \\ Q(u,v) \end{bmatrix}$ $\quad r = \begin{bmatrix} u \\ v \end{bmatrix}$ position of loop

circulation $\quad C_L(V) = \oint_L V \cdot dr = \oint P\,du + Q\,dv$

def. $\partial_u(Q)\big|_c \approx \dfrac{Q(c+\delta u) - Q(c)}{\delta u}$

For small rectangle L



$Q(c) - Q(a) = Q(a+\delta u) - Q(a)$

$= \partial_u(Q)\cdot\delta u$

Contribution to $C_L(V)$ on the vertical edges - is

$Q(c) - Q(a) \approx \partial_u Q\,\delta u\,\delta v$

Horiz. Contribution. $\quad P(\overset{\text{below}}{b}) - P(d) = \overbrace{-\partial_v P\,\delta v\,\delta u}^{\text{neg.}}$ dir.

So $\displaystyle \oint P\,du + Q\,dv \approx \partial_u Q\,\delta u\,\delta v - \partial_v P\,\delta v\,\delta u$

$\displaystyle \underbrace{}_{\text{circulation}} \approx \underbrace{(\partial_u Q - \partial_v P)}_{\text{curl of vector field } V = \binom{P}{Q}}\,\delta A$

$\Rightarrow$ Curl of V $\approx$ local circulation per unit area.

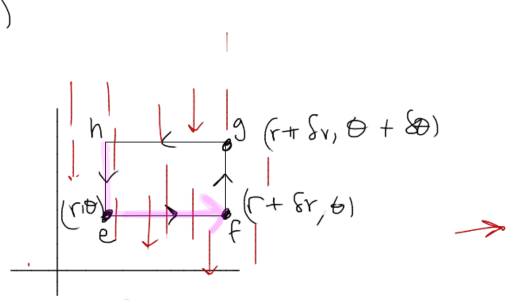
Holonomy in Flat Plane:

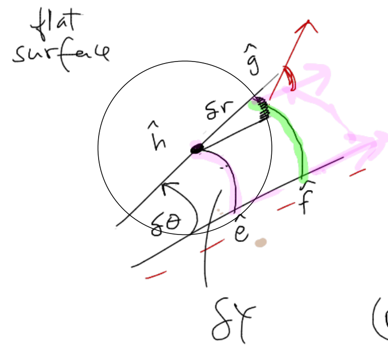
Recall: $R_u(\hat{K}) = \delta_{\hat{K}}(\angle u w_{\parallel})$ rotation of parallel transported $w_{\parallel}$ rel to u.

(ex) S = flat plane, polar coords $u = r$, $v = \theta$, $A = 1$, $B = r$

$$ds^2 = dr^2 + r^2 d\theta^2$$

(r, θ)





Rotatation of $w_{\parallel}$ arises b/c $\hat{g}\hat{f}$ is longer than $\hat{h}\hat{e}$ — by

Inner arc $r\,\delta\theta$

outer arc $(r+\delta r)\delta\theta$

$r\delta\theta + \underbrace{\delta r \delta\theta}_{\text{how much longer.}}$

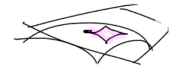
Take: $V = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$-\delta R_u(\hat{f}\hat{g}) \asymp \left[\frac{\text{arc}}{\text{radius}}\right] \asymp \left[\frac{\text{increase in side length}}{\text{orthogonal side}}\right] = \frac{\delta r \delta\theta}{\delta r} = \delta\theta$$

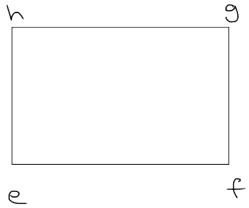
If $\delta Y \asymp B\delta v$, then $\delta(\delta Y) =$ increase from $u \mapsto u + \delta u$

$$\delta^2 Y = (\partial_u B \delta v)\delta u$$

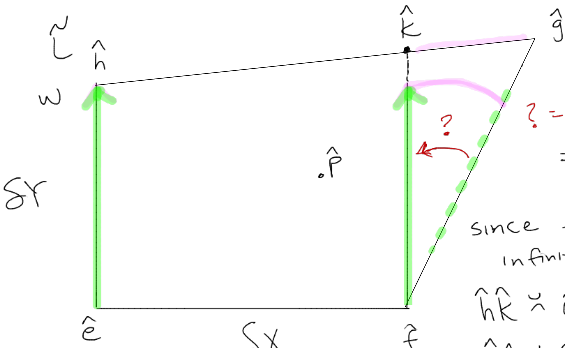
Now for general case



$L =$



(u,v) map plane



$B\delta v = \delta Y$

$? =$ angle $= \dfrac{\text{arc length}}{\text{radius}} \simeq \dfrac{\partial_v A \, \delta v \, \delta u}{B \delta v}$

since this is an infinitesimal ▭

$\hat{h}\hat{k} \simeq \hat{e}\hat{f} \simeq \delta X$

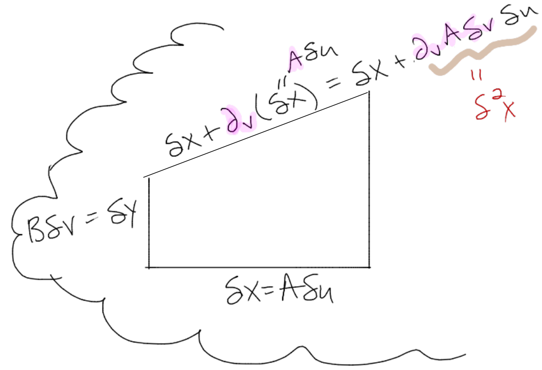
$\hat{k}\hat{g} \simeq \delta(\delta X) = \delta^2 X$

$\simeq$ arc length $\simeq \partial_v A \, \delta v \, \delta u$

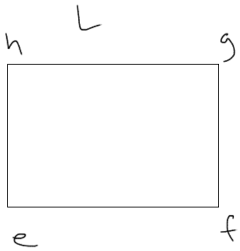
goal # 1

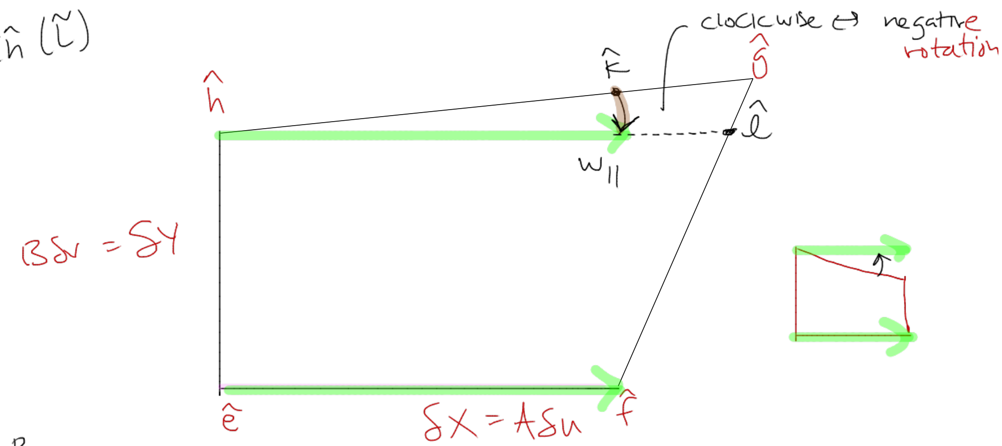
compute holonomy $R_{\partial \hat{f}}(\tilde{L}) = ? = \dfrac{\partial_v A}{B} \delta v$

① Parallel translate w to $\hat{f}$.



Similarly for $R_{\hat e \hat h}(\tilde L)$





Compute $R_{\hat e \hat h}(\tilde L) = \frac{\partial_u B}{A}\delta v$

(1) Parallel Translate w to $\hat h$

angle = $\frac{\text{arclength}}{\text{radius}}$ =

$$= \frac{(\partial_u B\,\delta u)\,\delta v}{A\,\delta u}$$

$\hat f\hat \ell \approx \hat e \hat h \approx \delta Y$

$\hat \ell \hat g \approx \delta(\delta Y) = \delta^2 Y$

$\approx$ arclength

$\approx (\partial_u B\,\delta u)\,\delta v$

Putting Together: $R(\tilde L) = \left(\frac{\partial_v(A)}{B}\right)\delta u - \left(\frac{\partial_u B}{A}\right)\delta v$

$$= \begin{bmatrix} \frac{\partial_v A}{B} \\ -\frac{\partial_u B}{A} \end{bmatrix} \cdot \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}$$

To get holonomy of general loop, integrate along its image

$$R(\tilde L) = C_V(L)$$

Get pointwise curvature, take limit as $\tilde L \to \hat p$.

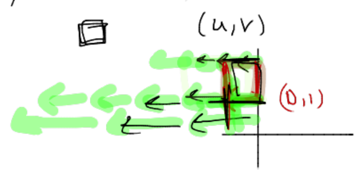
$$K(\hat p) = \lim_{\hat L \to \hat p} \frac{R(\tilde L)}{\delta \tilde A} \approx \frac{C_V(L)}{AB\,\delta A} = \frac{1}{AB}\left[\underbrace{\frac{C_V(L)}{\delta A}}_{\text{curl of } V.}\right]$$

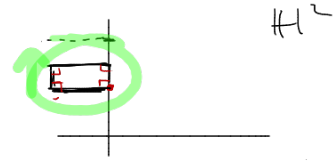
$$= \frac{1}{AB}\cdot(\text{curl of our } V) = \frac{1}{AB}\left(\partial_u\left(-\frac{\partial_u B}{A}\right) - \partial_v\left(\frac{\partial_v A}{B}\right)\right)$$

$$= -\frac{1}{AB}\left(\partial_v\left(\frac{\partial_v A}{B}\right) + \partial_u\left(\frac{\partial_u B}{A}\right)\right)$$

Metric Induced Vector Field from $\mathbb{H}^2$

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \qquad A = \frac{1}{y}, \; B = \frac{1}{y}$$





(x, y)

$$V = \begin{bmatrix} \dfrac{\partial_y\left(\frac{1}{y}\right)}{\left(\frac{1}{y}\right)} \\[2ex] \dfrac{\partial_x\left(\frac{1}{y}\right)}{\left(\frac{1}{y}\right)} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{-\frac{1}{y^2}}{\frac{1}{y}}\right) \\[2ex] 0 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{y} \\[2ex] 0 \end{bmatrix}$$

$\omega \times$

$\mathrm{Curl}(V) =$

$$\partial_x(0) - \partial_y\left(-\frac{1}{y}\right)$$

$$= \frac{-1}{y^2}$$

Homework:

Compute $K(p)$ for sphere $+1/R^2$ & $\mathbb{H}^2$. $-\frac{1}{R^2}$

using curl/circulation.