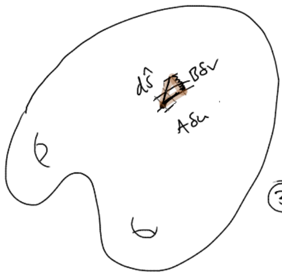
Metric Curvature Formula

$$K = \frac{-1}{AB}\left(\partial_v\left(\frac{\partial_v A}{B}\right) + \partial_u\left(\frac{\partial_u B}{A}\right)\right)$$

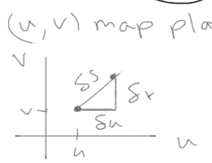
Construct Orthogonal Coord System on S.



① On surface @ $\hat{p}$, draw non-intersecting curves (u-curves) thru, near $\hat{p}$ — "number" them differentiably
u-coord

② Form orthogonal trajectories, similarly (v-coord)
v-curves

③ let X = measure dist along v-curves on S (hor)
Y " " " " " " " (vert)



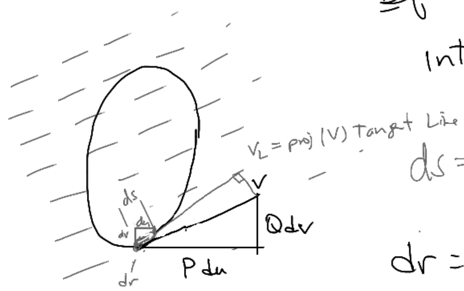
④ $\delta X = A\,\delta u$, $\delta Y = B\,\delta v$

$ds^2 = A^2 du^2 + B^2 dv^2$

Tool: Circulation of a vector field around a loop

$V = \begin{bmatrix} P(u,v) \\ Q(u,v) \end{bmatrix}$ vector field in (u,v) plane $\qquad r = \begin{bmatrix} u \\ v \end{bmatrix}$ position of loop

Def Circulation $C_L(V)$ of V around L: integral of component of V in direction of L.



$ds = |dr|$

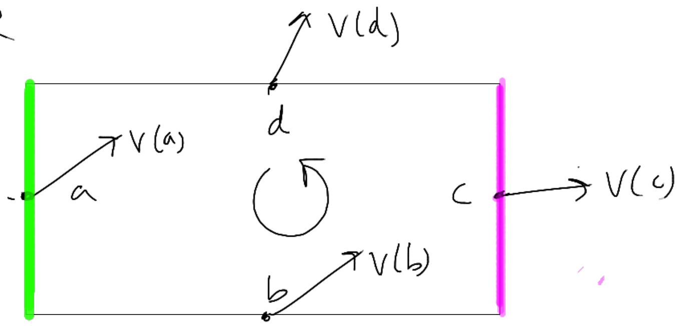
$dr = \begin{pmatrix} du \\ dv \end{pmatrix}$

$$\boxed{C_L(V) = \oint_L V \cdot dr = \int V_L\, ds = \oint_L P\,du + Q\,dv}$$

---

ex For small rectangle R (not $\mathbb{R}$), calculate $C_R(V)$, via Riemann Sum

R



Contribution to $C_R(V)$ from | is $Q(c)\,\delta v$ & from $-Q(a)\,\delta v$

$\rightsquigarrow$ Verti: $(Q(c) - Q(a))\,\delta v$

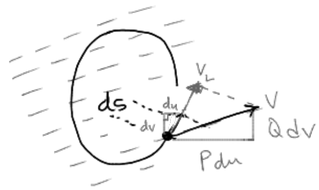
Horizontal: $(P(b) - P(d))\,\delta u = -(P(d) - P(b))\,\delta u$

Total $\approx$ $\partial_u Q\, \delta u\, \delta v - \partial_v P\, \delta v\, \delta u$

Tool: the Circulation of a vector Field around a loop

Let $V = \begin{bmatrix} P(u,v) \\ Q(u,v) \end{bmatrix}$ be a vector field in (u,v) plane & $r = \begin{bmatrix} u \\ v \end{bmatrix}$ position vector of loop

Def'n: Circulation $C_L(V)$ of V around L is integral of component of V in the direction of L.

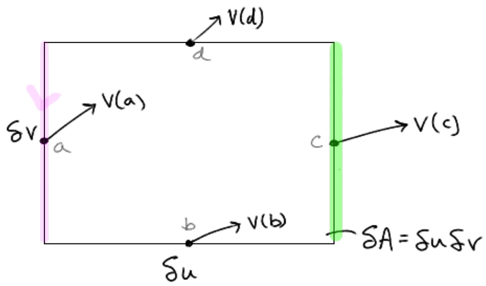


$ds = |dr|$

$dr = \begin{pmatrix} du \\ dv \end{pmatrix}$

$$C_L(V) = \oint_L V \cdot dr = \oint V_L ds = \oint_L P du + Q dv$$

For small rectangle R, calculate $C_R(V)$ (via Riemann Sum)



Contribution to $C_R(V)$ from ↑ is $\approx Q(c)\delta v$

from ↓ is $\approx -Q(a)\delta v$ ... total: $(Q(c) - Q(a))\delta v$

Same for horizontal: $(P(b) - P(a))\delta u$

$\Rightarrow C_R(V) \approx (Q(c) - Q(a))\delta v +$

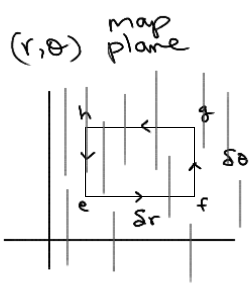
Holonomy In Flat Plane

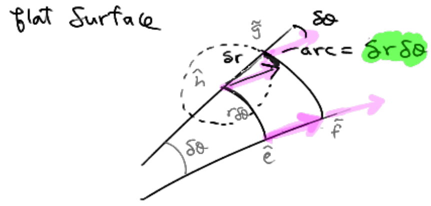
Recall: Holonomy Along Curve $= R_u(\hat{K}) \equiv \delta_{\hat{K}}(\angle u w_{\parallel})$ rotation of parallel t'd $w_{\parallel}$ rel. to u.

- when K = loop $R_u(\hat{L}) = R(\hat{L})$ becomes indep of u

ex S = flat plane w/ polar coords $u = r,\ v = \theta,\ A = 1,\ B = r$

$\rightsquigarrow\ ds^2 = dr^2 + r^2 d\theta^2$





Rotation of $w_{\parallel}$ arises b/c $\hat{g}\hat{f}$ is longer than $\hat{h}\hat{e}$ — by

$r\,\delta\theta \longmapsto (r+\delta r)\,\delta\theta$

$= r\,\delta\theta + \underbrace{\delta r\,\delta\theta}_{\text{extra length}}$

so $\delta R_u(\hat{h}\hat{e}) = \delta\theta$

and $-\delta R_u(\hat{f}\hat{g}) \asymp \dfrac{\text{arc}}{\text{radius}} \asymp \dfrac{\text{increase in side length}}{\text{orthogonal side}} = \dfrac{\delta r\,\delta\theta}{\delta r} = \delta\theta$

More generally, if $\delta Y \asymp B\,\delta v$ then $\delta^2 Y = \delta(\delta Y)$ $\left\{\begin{array}{l}\text{increase from}\\ u \mapsto u+\delta u\end{array}\right\}$

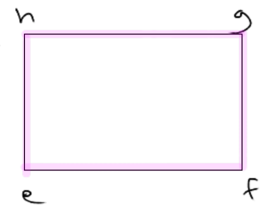
is $\delta Y \asymp [\partial_u B\,\delta v]\,\delta u$

Note on map-plane (left) $\delta R_u(\text{edge}) = 0\,\delta r + (-1)\,\delta\theta = \begin{pmatrix}0\\-1\end{pmatrix}\cdot\begin{pmatrix}\delta r\\ \delta\theta\end{pmatrix}$

holonomy of loop = circulation of V around loop in map.

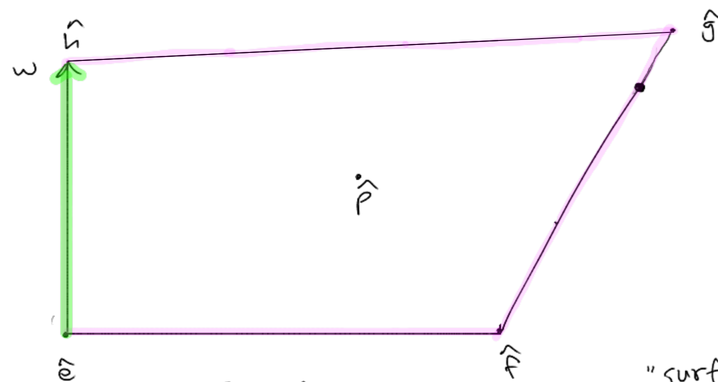
Holonomy as Circulation of Metric Induced Vector Field in Map

$L =$



(u,v)-map

$\hat{L}$



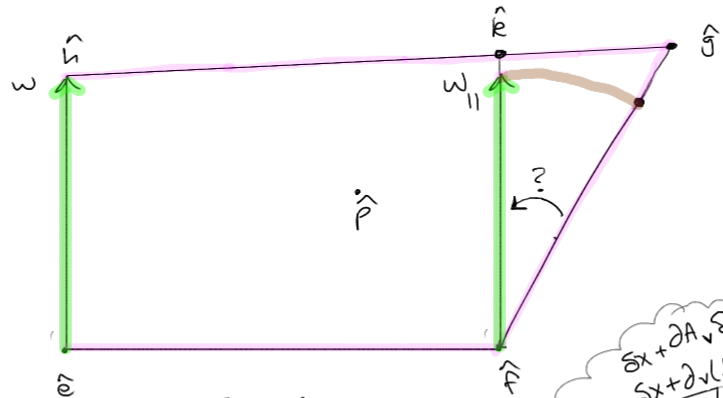
"surface"

actually since its curved, this is the local projection to tangent plane

Compute holonomy $R_{\hat{e}\hat{f}}(\hat{L})$

① Parallel Translate w to $\hat{f}$. (Recall, this is defined as: 'project to $T_{\hat{p}}(S)$ - translate Euclideanly - proj. back')

$\hat{L}$



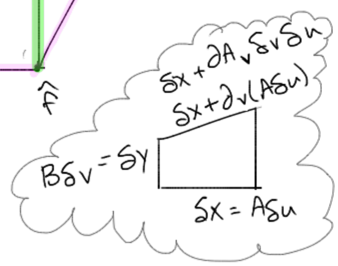
$$? = \text{angle} = \frac{\text{arc length}}{\text{radius}}$$

(since this is infintessimal $\square$)

$\hat{h}\hat{k} \asymp \hat{e}\hat{f} \asymp \delta X$

$\hat{k}\hat{g} \asymp \delta(\delta X) = \delta^2 X$

$\asymp$ arc length



$\delta^2 X \asymp [\partial A_v \delta v]\delta u$

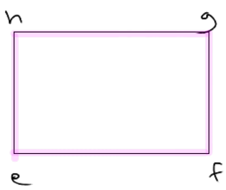
radius $\asymp \delta Y$

$$? = \frac{[\partial A_v \delta v]\delta u}{B\delta v} = \left(\frac{\partial A_v}{B}\right)\delta u$$

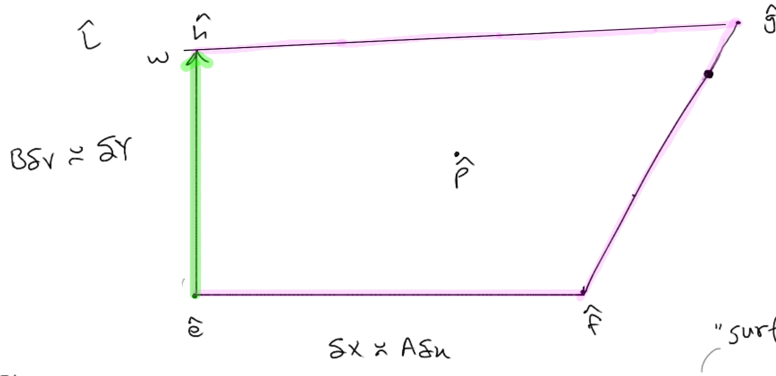
$R_{\hat{e}\hat{f}}(\hat{L})$ defined wrt fiducial vector field, which here is the (u,v)-curves

$$\boxed{R_{\hat{e}\hat{f}}(\hat{L}) = \left(\frac{\partial A_v}{B}\right)\delta u}$$

Similar for $R_{\hat{e}\hat{h}}(\tilde{L})$

$L =$ 

(u,v)-map

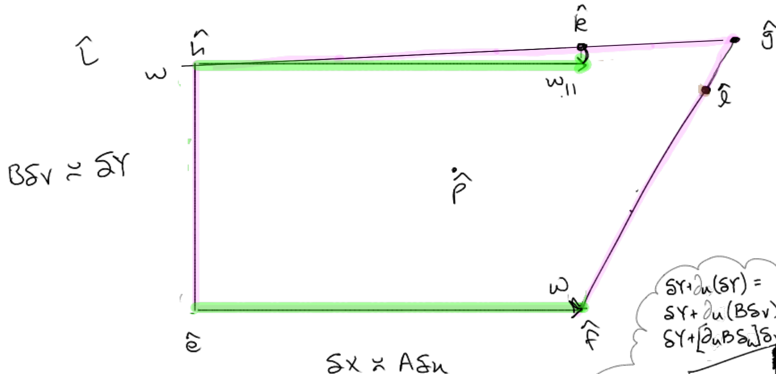


Compute holonomy $R_{\hat{e}\hat{h}}(\hat{L})$

① Parallel Translate w to $\hat{h}$. (Recall, this is defined as: 'project to $T_{\hat{p}}(S)$ - translate Euclideanly - proj. back')

"surface"

actually since its curved, this is the local projection to tangent plane



(rotation wrt (u,v) curves)

$$? = \text{angle} = \frac{\text{arc length}}{\text{radius}}$$

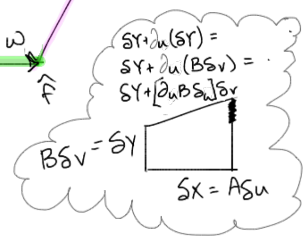
(since this is infintessimal □)

$\hat{f}\hat{\ell} \simeq \hat{e}\hat{h} \simeq \delta Y$

$\hat{\ell}\hat{g} \simeq \delta(\delta Y) = \delta^2 Y$

$\simeq$ arc length

$\delta^2 Y \simeq [\partial_u B\, \delta_u]\delta v$



radius $\simeq \delta X$

negative (clockwise)

$$? = \frac{[\partial B_u \delta_u]\delta v}{A\delta u} = \left(\frac{\partial B_u}{A}\right)\delta v$$

$$\boxed{R_{\hat{e}\hat{h}}(\hat{L}) = -\left(\frac{\partial A_v}{B}\right)\delta u}$$

Putting together: (linearity)

$$R(\hat{L}) = \left(\frac{\partial_v A}{B}\right)\delta u - \left(\frac{\partial_u B}{A}\right)\delta v = \begin{bmatrix} \frac{\partial_v A}{B} \\ -\frac{\partial_u B}{A} \end{bmatrix} \cdot \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}$$

to get holonomy of a general loop $\hat{L}$, integrate along its image in (u,v)-map.

$$\oint V \cdot \begin{bmatrix} \delta u \\ \delta v \end{bmatrix} = C_V(L)$$

$$R(\hat{L}) = C_V(L)$$

Pointwise: take limit $\hat{L} \to \hat{p}$

$$K(\hat{p}) = \lim_{\hat{L}\to\hat{p}} \frac{R(\hat{L})}{\delta\hat{A}} \approx \frac{C_V(L)}{AB\,\delta A} = \frac{1}{AB}\Bigg[\underbrace{\frac{C_V(L)}{\delta A}}_{\substack{\text{local circulation} \\ \text{per unit area} \\ = \text{curl of } V}}\Bigg]$$

$$K(\hat{p}) = \frac{1}{AB}\left\{ \text{curl of } \begin{bmatrix} \partial_v A / B \\ -\partial_u B / A \end{bmatrix} \right\}$$

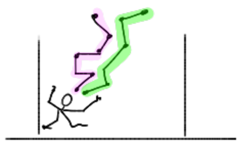
$$= \frac{1}{AB}\left\{ \partial_u\left[\frac{-\partial_u B}{A}\right] - \partial_v\left[\frac{\partial_v A}{B}\right] \right\}$$

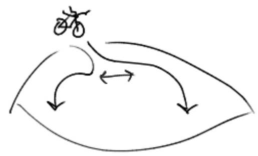
$$= -\frac{1}{AB}\left(\partial_v\left[\frac{\partial_v A}{B}\right] + \partial_u\left[\frac{\partial_u B}{A}\right] \right)$$

the Jacobi Equation

"curvature as a force b/w neighboring geodesics"

- key: separation of neighboring unit-speed geodesics, their "relative acceleration"





Fundamental Eq'n governs this ↑

- Eq'n of geodesic deviation
- Jacobi Equation (1837)

two nearby geodesics enter

$+$ attract

$-$ repel

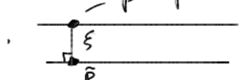
same source different direction

↑ may intersect @ conjugate point

never again shall meet

ex: Euclidean Plane

p = position

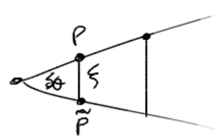


velocity
$v = \dot{p}$, $\tilde{v} = \dot{\tilde{p}}$

perpendicular : $\xi = \overrightarrow{p\tilde{p}}$

$\dot{\xi} = 0$

so $\ddot{\xi} = 0$



$\dot{\xi} = |\delta\theta|$

since $\xi = r|\delta\theta|$

$\dot{r} = 1$ since unit speed

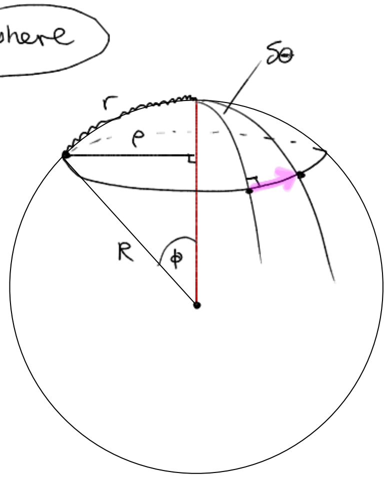
so $\ddot{\xi} = 0$

def'n

$$\ddot{\xi} = \frac{d(\delta v)}{dt}$$

---

ex Sphere



Two particles launched from N @ unit speed, $(t = R\phi)$ $\delta\theta$ apart.

p(t) lies on circle w/ intrinsic radius $= r$

$\rho =$ extrinsic radius $= R\sin\phi$

$= R\sin\left(\frac{t}{R}\right)$

So $|\xi| = \rho\,\delta\theta = R\,\delta\theta\,\sin\left(\frac{t}{R}\right)$

$$\boxed{\ddot{\xi} = -K\xi}$$

$$\frac{d\xi}{dt} = R\,\delta\theta\cos\left(\frac{t}{R}\right)\cdot\frac{1}{R} = \delta\theta\cos\left(\frac{t}{R}\right)$$

$$\frac{d^2\xi}{dt^2} = \delta\theta\left(-\sin\left(\frac{t}{R}\right)\right)\left(\frac{1}{R}\right) = -\frac{1}{R}\delta\theta\sin\left(\frac{t}{R}\right) = \underbrace{-\frac{1}{R^2}}_{-K}\,\underbrace{R\,\delta\theta\sin\left(\frac{t}{R}\right)}_{\xi}$$

Note:

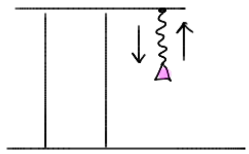
$\ddot{\xi} = -k\xi$

is an O.D.E.

(Harmonic Oscillator)

w/ sol'n:

$\xi = \xi_0 \sin(\sqrt{k}\, t)$



Hooke's Law:

Force pulling weight back up is proportional to displacement from equil.

(k = stiffness)

(ex) on sphere

geodesics initially diverge

$\sin(\epsilon) \simeq \epsilon$

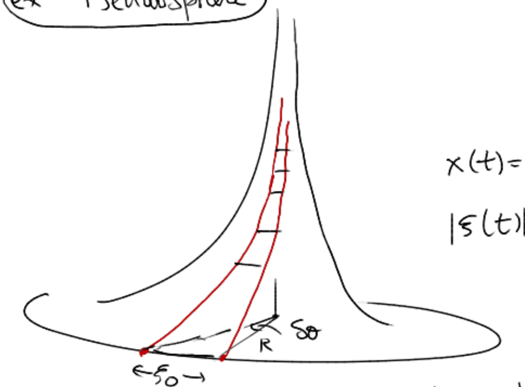
as $r \to 0$ $\xi = R\,\delta\theta \sin\left(\frac{r}{R}\right)$

$\simeq R\,\delta\theta \cdot \frac{r}{R} = r\,\delta\theta$

Despite particles moving freely on surface, the curvature acts as a attractive force (like a spring) which slows their separation.

At South Pole, particles meet w/ maximal rel. velocity $\leftrightarrow$ weight passing equilibrium w/ max. velocity.

ex Pseudosphere



$x(t)$ = dist b/w particles & axis @ time t

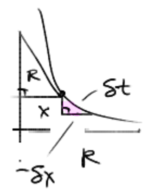
$$|\xi(t)| = x(t)\cdot\delta\theta \simeq \frac{|\xi_0|}{R}\cdot x(t) \qquad (*)$$

particles leave rim w/ unit speed: $\sigma = t$ arc length = time

Rim is dist. R from axis. Particles are initially pointing toward axis.

$|\xi_0| \simeq R\,\delta\theta$

since by construction (def)



$$\boxed{-\frac{dx}{dt} = \frac{x}{R}} \qquad \text{or} \qquad -\dot{x} = \frac{x}{R}$$

then $(*)$ $\Rightarrow$ $\dot{\xi} = \frac{|\xi_0|}{R}\dot{x} = \frac{|\xi_0|}{R}\cdot\frac{-x}{R} = -\frac{1}{R}\xi$

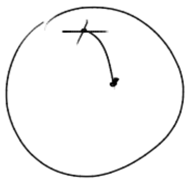
So $\dot{\xi}_0 = \frac{|\xi_0|}{R}$ and in the absense of any force the particles would meet in time $t = R$

they don't:

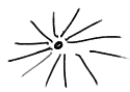
$$\ddot{\xi} = -\frac{1}{R}\dot{\xi} = +\left(\frac{1}{R^2}\right)\xi = -K\xi$$

General Case of Jacobi Equation

Geodesic Polar Coords

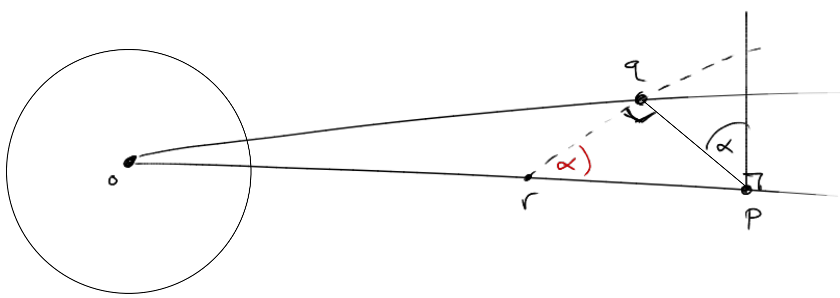


Pick a pt as $(0,0)$. Launch geodesics. Every pt in a suff. small nbhd lies on exactly one geodesic.



Gauss's lemma: Launched geodesics form a geodesic circle $K(\sigma)$ w/ intrinsic radius σ, cutting its radii @ right angles

Let $p, q \in K(R)$



If $pq \perp op$ & oq done.

ABWOC:

1. $\angle opq = \pi/2 - \alpha$
2. take geodesic thru $q \perp pq$ $\Rightarrow \angle qrp = \alpha$
3. $rq \asymp rp \cos\alpha$
4. $\text{length}(o-r-q) = or + rq$ $\asymp (R - rp) + rp\cos\alpha$ $\asymp R - rp(1-\cos\alpha)$ $< R$
5. But oq is a geodesic, (unique) & no shorter path exists, so $pq \perp op$ & oq
6. thus (t,θ) are orthogonal coords! $d\hat{s}^2 = dt^2 + \rho^2(t,\theta)\,d\theta^2$
7. If $\delta\theta$ = angular separation of two geodesics: $|\xi| \asymp \rho\,\delta\theta \Rightarrow |\ddot{\xi}| \asymp \ddot{\rho}\,\delta\theta$
8. sub: $u=t$, $v=\theta$, $A=1$, $B=\rho$ into curvature formula

$$K(p) = -\frac{1}{\rho}\left(\partial_\theta\left(\frac{\partial_\theta(1)}{\rho}\right) + \partial_t\left(\frac{\partial_t(\rho)}{1}\right)\right)$$

$$= -\frac{1}{\rho}\left(\ddot{\rho}\right) = -\frac{\ddot{\rho}}{\rho} \overset{7}{=} -\frac{\frac{|\ddot{\xi}|}{\delta\theta}}{\frac{|\xi|}{\delta\theta}} = -\frac{\ddot{\xi}}{\xi} \Rightarrow \boxed{\ddot{\xi} = -K\xi}$$

Note: this can alternatively be proven via: 'relative acceleration = holonomy of velocity'