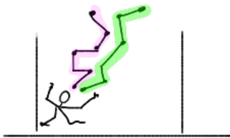
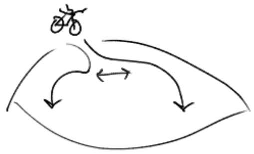
The Jacobi Equation

"curvature as a force b/w neighboring geodesics"

- key: separation of neighboring unit-speed geodesics, their "relative acceleration"





Fundamental Eq'n governs this ↑

- Eq'n of geodesic deviation
- Jacobi Equation (1837)

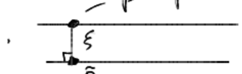
two nearby geodesics enter

same source different direction

may intersect @ conjugate point

never again shall meet

ex: Euclidean Plane

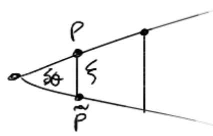


velocity $v = \dot{p}$, $\tilde{v} = \dot{\tilde{p}}$

perpendicular : $\xi = \overrightarrow{p\tilde{p}}$

$\dot{\xi} = 0$

so $\ddot{\xi} = 0$



$\dot{\xi} = |\delta\theta|$

since $\xi = r|\delta\theta|$

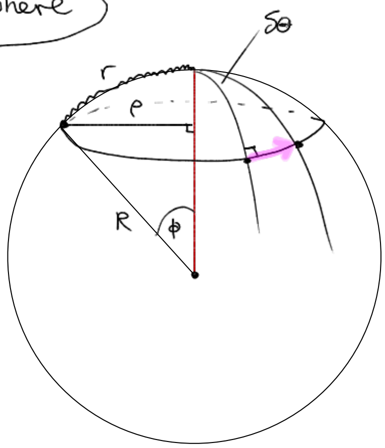
$\dot{r} = 1$ since unit speed

so $\ddot{\xi} = 0$

def'n

$$\ddot{\xi} = \frac{d(\delta v)}{dt}$$

ex Sphere



Two particles launched from N @ unit speed, $(t = R\phi)$ $\delta\theta$ apart.

$p(t)$ lies on circle w/ intrinsic radius $= r$

$\rho =$ extrinsic radius $= R\sin\phi$

$= R\sin\left(\frac{t}{R}\right)$

So $|\xi| = \rho\,\delta\theta = R\,\delta\theta\sin\left(\frac{t}{R}\right)$

$$\boxed{\ddot{\xi} = -K\xi}$$

$$\frac{d\xi}{dt} = R\,\delta\theta\cos\left(\frac{t}{R}\right)\cdot\frac{1}{R} = \delta\theta\cos\left(\frac{t}{R}\right)$$

$$\frac{d^2\xi}{dt} = \delta\theta\left(-\sin\left(\frac{t}{R}\right)\right)\left(\frac{1}{R}\right) = -\frac{1}{R}\delta\theta\sin\left(\frac{t}{R}\right) = \underbrace{-\frac{1}{R^2}}_{-K}\;\underbrace{R\,\delta\theta\sin\left(\frac{t}{R}\right)}_{\xi}$$

Note:

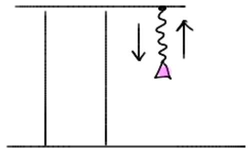
$$\ddot{\xi} = -k\xi$$

is an O.D.E.

(Harmonic Oscillator)

w/ sol'n:

$$\xi = \xi_0 \sin(\sqrt{k}\, t)$$



Hooke's Law:

Force pulling weight back up is proportional to displacement from equil. (k = stiffness)

ex on sphere

geodesics initially diverge

$\sin(\epsilon) \simeq \epsilon$

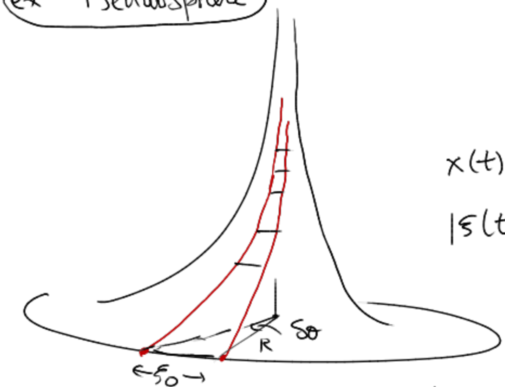
as $r \to 0$ $\quad \xi = R\,\delta\theta \sin\left(\frac{r}{R}\right)$

$$\simeq R\,\delta\theta \cdot \frac{r}{R} = r\,\delta\theta$$

Despite particles moving freely on surface, the curvature acts as a attractive force (like a spring) which slows their separation.

At South Pole, particles meet w/ maximal rel. velocity $\leftrightarrow$ weight passing equilibrium w/ max. velocity.

ex Pseudosphere



$x(t)$ = dist b/w particles & axis @ time t

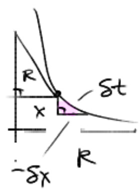
$$|\xi(t)| = x(t)\cdot\delta\theta \simeq \frac{|\xi_0|}{R}\cdot x(t) \qquad (*)$$

particles leave rim w/ unit speed: $\sigma = t$
arc length = time

Rim is dist. R from axis. Particles are initially pointing toward axis.

$|\xi_0| \simeq R\,\delta\theta$

since by construction (def)



$$\boxed{-\frac{dx}{dt} = \frac{x}{R}} \qquad \text{or} \qquad -\dot{x} = \frac{x}{R}$$

then (*) $\Rightarrow$ $\dot{\xi} = \frac{|\xi_0|}{R}\dot{x} = \frac{|\xi_0|}{R}\cdot\frac{-x}{R} = -\frac{1}{R}\xi$

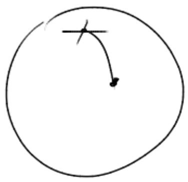
So $\dot{\xi}_0 = \frac{|\xi_0|}{R}$ and in the absense of any force the particles would meet in time $t = R$

they don't:

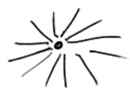
$$\ddot{\xi} = -\frac{1}{R}\dot{\xi} = +\left(\frac{1}{R^2}\right)\xi = -K\xi$$

General Case of Jacobi Equation

Geodesic Polar Coords



Pick a pt as $(0,0)$. Launch geodesics. Every pt in a suff. small nbhd lies on exactly one geodesic.

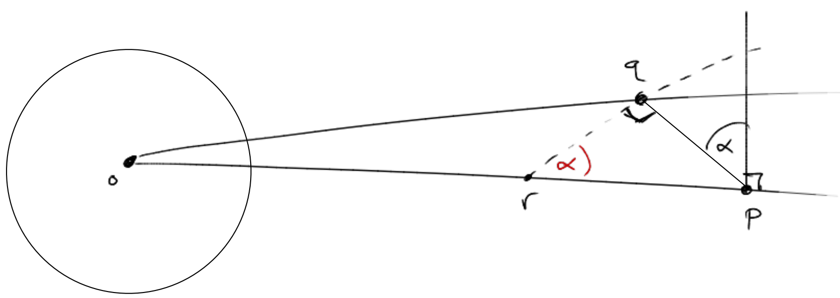


Gauss's lemma: Launched geodesics form a geodesic circle $K(\sigma)$ w/ intrinsic radius σ, cutting its radii @ right angles

Let $p, q \in K(R)$

If $pq \perp op$ & oq done.

ABWOC:



1. $\angle opq = \pi/2 - \alpha$
2. take geodesic thru $q \perp pq$ $\Rightarrow \angle qrp = \alpha$
3. $rq \asymp rp\cos\alpha$
4. $\text{length}(o-r-q) = or + rq$
   $\asymp (R - rp) + rp\cos\alpha$
   $\asymp R - rp(1-\cos\alpha)$
   $< R$
5. But oq is a geodesic, (unique) & no shorter path exists, so $pq \perp op$ & oq
6. thus (t, θ) are orthogonal coords!

$$d\hat{s}^2 = dt^2 + \rho^2(t,\theta)\,d\theta^2$$

7. If $\delta\theta$ = angular separation of two geodesics: $|\xi| \asymp \rho\,\delta\theta \Rightarrow |\ddot{\xi}| \asymp \ddot{\rho}\,\delta\theta$

8. sub: $u = t$, $v = \theta$, $A = 1$, $B = \rho$ into curvature formula

$$K(p) = -\frac{1}{\rho}\left(\partial_\theta\left(\frac{\partial_\theta(1)}{\rho}\right) + \partial_t\left(\frac{\partial_t(\rho)}{1}\right)\right)$$

$$= -\frac{1}{\rho}\left(\ddot{\rho}\right) = -\frac{\ddot{\rho}}{\rho} \overset{7}{=} -\frac{\frac{|\ddot{\xi}|}{\delta\theta}}{\frac{|\xi|}{\delta\theta}} = -\frac{\ddot{\xi}}{\xi} \Rightarrow \boxed{\ddot{\xi} = -K\xi}$$

Note: this can alternatively be proven via: 'relative acceleration = holonomy of velocity'