Notes on Needham's Derivation of the Riemann Curvature Tensor

Needham defines the covariant derivative of a vector field w in the direction of a vector (field) $\mathbf{v}$ as

Covariant Derivative Definition

$$\nabla_{\mathbf{v}}\mathbf{w} \asymp \frac{\mathbf{w}_{||}(q \to p) - \mathbf{w}(p)}{\epsilon}$$

where q is a point "infinitesimally" close to p in the direction of $\mathbf{v}$ and $\mathbf{w}_{||}(q \to p)$ is the parallel transport of $\mathbf{w}(q)$ back to p along the geodesic from q to p. Needham also indicates this derivative represents how much the vector $\mathbf{w}(q)$ differs from its original position $\mathbf{w}(p)$ after being moved back to p via parallel transport. It is implied, then, that the point at which this derivative is evaluated is p

$$\nabla_{\mathbf{v}}(p)\mathbf{w} \asymp \frac{\mathbf{w}_{||}(q \to p) - \mathbf{w}(p)}{\epsilon}.$$

The Vector Holonomy

Recall, the holonomy

$$R_U(K) \equiv \delta_K(\angle \mathbf{w}_{||})$$

measures the angle **FROM** a fixed fiducial vector field U - **TO** the parallel transported vector $\mathbf{w}_{||}$ along K. If we choose the fiducial vector field U to be the vector field w itself, the holonomy measures the angle from w to its parallel transport $\mathbf{w}_{||}$ along the infinitesimal curve K from p to q. This is exactly what the covariant derivative measures, except for a negative sign.

The Negative Holonomy

The negative vector holonomy $-R_U(K)$ measures the opposite - the angle **FROM** the parallel transported $\mathbf{w}_{||}$ - **TO** the fiducial vector field. This is exactly what the covariant derivative measures in the infinitesimal case! That is,

$$\nabla_{\mathbf{v}}\mathbf{w} \asymp \frac{\mathbf{w}_{||}(q \to p) - \mathbf{w}(p)}{\epsilon} - R_{\mathbf{w}}K).$$

Computing Holonomy Along the 5 Segments

By the Mean Value Theorem, the negative vector holonomy along K (being a continuous, differentiable function measuing the change in w from the vector field - to the PT'd vector along K) is ultimatly

$$-\delta_{oa}(\angle \mathbf{w}_{||} w) \asymp \mathbf{w}(o^*) - \mathbf{w}_{||}(o^*) \asymp \nabla_{\mathbf{v}}\mathbf{w}(o^*)$$

where $K = oa$ is in the direction of $\mathbf{v}$ and o^* lies between o and a.

The full negative holonomy of the loop can be computed as the sum of the negative vector holonomies along each of the 5 segments that make up K.

Putting it all Together

Thus, the covariant derivative can be computed as the sum of the negative vector holonomies along each segment of K:

$$-R_{\mathbf{w}}(L) \asymp -R_{\mathbf{w}}(oa) - R_{\mathbf{w}}(ab) - R_{\mathbf{w}}(bq) - R_{\mathbf{w}}(qp) - R_{\mathbf{w}}(po)$$

$$\asymp \nabla_{\delta\mathbf{u}\,u}\mathbf{w}(o^*) + \nabla_{\delta\mathbf{v}\,\mathbf{v}}\mathbf{w}(a^*) + \nabla_{c}\mathbf{w}(c^*) - \nabla_{\delta\mathbf{u}\,u}\mathbf{w}(q^*) - \nabla_{\delta\mathbf{v}\,\mathbf{v}}\mathbf{w}(p^*)$$

where a^*, b^*, c^*, p^*, q^* are points along each segment of K coming from the Mean Value Theorem. The p^* point lies between p and o, so we can approximate it as the midpoint of op. Similarly for the other * points, q^* is approximately the midpoint of pq, c^* approximately the midpoint of bq and a^* approximately the midpoint of ab.

The p^* point lies in a segment traversed in the $-v$ direction, so we have a negative sign in front of that term. Similarly for the q^* point. Thus, we have

$$-R_{\mathbf{w}}(L) \asymp \delta\mathbf{u}\nabla_{\mathbf{u}}\mathbf{w}(a^*) + \delta\mathbf{v}\nabla_{\mathbf{v}}\mathbf{w}(b^*) + \nabla_{c}\mathbf{w}(c^*) - \delta\mathbf{u}\nabla_{\mathbf{u}}\mathbf{w}(q^*) - \delta\mathbf{v}\nabla_{\mathbf{v}}\mathbf{w}(p^*)$$

we prepare to collect the terms with $\delta\mathbf{u}$ and $\delta\mathbf{v}$ we collect like terms with color coding:

$$-R_{\mathbf{w}}(L) \asymp \boxed{\delta\mathbf{u}\nabla_{\mathbf{u}}\mathbf{w}(a^*)}\boxed{+\delta\mathbf{v}\nabla_{\mathbf{v}}\mathbf{w}(b^*)}\boxed{+\nabla_c\mathbf{w}(c^*)}\boxed{-\delta\mathbf{u}\nabla_{\mathbf{u}}\mathbf{w}(q^*)}\boxed{+\delta\mathbf{v}\nabla_{\mathbf{v}}\mathbf{w}(p^*)}$$

Notice that a^* and q^* are both approximately the midpoint of ab and pq, respectively, so we can combine those terms. Similarly for b^* and p^*. This gives us

$$-R_{\mathbf{w}}(L) \asymp \boxed{\delta\mathbf{v}\left(\nabla_{\mathbf{v}}\mathbf{w}(b^*) - \nabla_{\mathbf{v}}\mathbf{w}(p^*)\right)} - \boxed{\delta\mathbf{u}\left(\nabla_{\mathbf{u}}\mathbf{w}(q^*) - \nabla_{\mathbf{u}}\mathbf{w}(a^*)\right)} + \boxed{\nabla_c\mathbf{w}(c^*)}$$

Recalling the vector c is the scaled commutator

$$c \asymp \delta\mathbf{u}\,\delta\mathbf{v}[\mathbf{v},\mathbf{u}] \asymp -\delta\mathbf{u}\,\delta\mathbf{v}[\mathbf{u},\mathbf{v}]$$

so we have

$$-R_{\mathbf{w}}(L) \asymp \boxed{\delta\mathbf{v}\left(\nabla_{\mathbf{v}}\mathbf{w}(b^*) - \nabla_{\mathbf{v}}\mathbf{w}(p^*)\right)} - \boxed{\delta\mathbf{u}\left(\nabla_{\mathbf{u}}\mathbf{w}(q^*) - \nabla_{\mathbf{u}}\mathbf{w}(a^*)\right)} - \boxed{\nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w}(c^*)}$$

$$-R_{\mathbf{w}}(L) \asymp \delta\mathbf{v}\boxed{\left(\nabla_{\mathbf{v}}\mathbf{w}(b^*) - \nabla_{\mathbf{v}}\mathbf{w}(p^*)\right)} + \delta\mathbf{u}\boxed{\left(\nabla_{\mathbf{u}}\mathbf{w}(q^*) - \nabla_{\mathbf{u}}\mathbf{w}(a^*)\right)} - \delta\mathbf{v}\,\delta\mathbf{u}\nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w}(c^*).$$

Applying the Mean Value Theorem again to the expressions boxed above, we get

$$-R_{\mathbf{w}}(L) \asymp \delta\mathbf{u}\,\delta\mathbf{v}\nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w}(r^*) - \delta\mathbf{v}\,\delta\mathbf{u}\nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w}(s^*) - \delta\mathbf{u}\,\delta\mathbf{v}\nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w}(c^*)$$

where r^* is between a^* and q^* and s^* is between b^* and p^*. Both of these points are approximately the center of the loop L, so we can combine them into a single point l^* at the center of L. The point c^* too becomes arbitrarily close to l^* in the limit thus

$$-R_{\mathbf{w}}(L) \asymp \delta\mathbf{u}\,\delta\mathbf{v}\left(\nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w}(l^*) - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w}(l^*) - \nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w}(l^*)\right).$$

Finally, dividing both sides by the area of the loop $\delta\mathbf{u}\,\delta\mathbf{v}$ taking the limit as the area goes to zero, we see $l^* \to o$ and

$$\lim_{L\to o}\frac{-R_{\mathbf{w}}(L)}{\mathrm{Area}(L)} = \nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w} - \nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w}.$$

We can write this more compactly using the commutator notation,

$$[u,v] = uv - vu$$

as $u = \nabla_{\mathbf{u}}$ and $v = \nabla_{\mathbf{v}}$ and reverting back to ultimate equality, and defining the Riemann Curvature Tensor as:

$$\boxed{\mathcal{R}(\mathbf{u},\mathbf{v})\mathbf{w} \asymp \frac{-R_{\mathbf{w}}(L)}{\mathrm{Area}(L)} \asymp [\nabla_{\mathbf{u}},\nabla_{\mathbf{v}}]\mathbf{w} - \nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w}.}$$