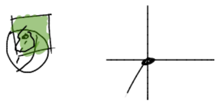
Riemann's Curvature Tensor

Setting: geometric 3-manifolds (and higher)



⓪ Recall $K(p) = \lim_{\Delta \to p} \frac{\mathcal{E}(\Delta)}{A(\Delta)}$ a scalar quantity (isn't enough to capture the complexity of 3-mflds geometry.)

Even slicing the M^3 into coord planes & computing curvature of those surfaces.

① The curvatures of a 3-mfld

① sectional curvature:

- From $p \in M^3$, construct T_pM ($\approx \mathbb{R}^3$), take any 2-D subspace σ_p. this is the tangent space of a sub-surface $\Sigma_p \subset M^3$.
- The sectional curvature $K(\sigma_p)$ is Gaussian Curvature of Σ_p.

② scalar curvature:

- sum of "some" sectional curvatures (see: Gromov IHES - 2019)
- sum of Eigenvalues of Ricci Tensor
- control volume
  - \+ scalar curvature ⇒ volume of ball is smaller than the corresp. Euclid. Ball
  - − scalar curvature ⇒ " " " " larger than " " " "

$S^1 \times$ ⓪

$S^1 \times D =$ solid ⓪

$S^1 \times S^1 =$ ⓪

- Ball in $S^1 \times \mathbb{H}^2$ has volume "very close" to ball in $\mathbb{E}^3$.

(+) × (−)

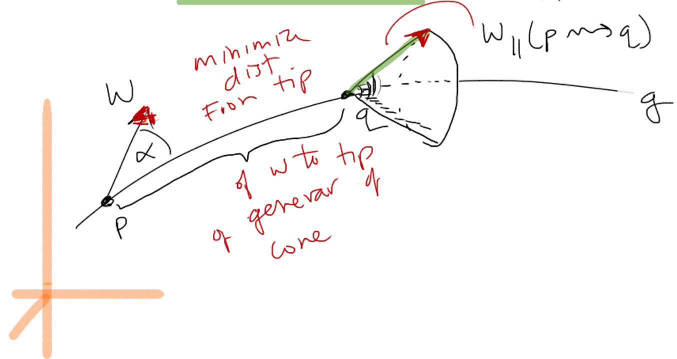
③ Ricci Curvature:
- (0,2) - tensor
- controls vol. rel to $\mathbb{E}^n$.

All the above are derived from Riemann Tensor.

Parallel Transport

Holonomy — Covariant Derivative

How to Parallel Transport in M^3



Covariant Derivative: ∇_v

{ Intrinsic - so we do not try to generalize our previous $D_v w = \text{proj.}(D_v w)$
$= \nabla_v w - (n \cdot D_v w) \cdot n$

Rate of change of vector field w as move from p in direction v
(moving along $\varepsilon \vec{v} = \vec{\varepsilon}$)

How much the new vector $w(q)$ has changed from OG $w(p)$ —
we P.T. it back as $w_{\|}(q \rightsquigarrow p)$

$$\boxed{\nabla_v w \approx \frac{w_{\|}(q \rightsquigarrow p) - w(p)}{\varepsilon}}$$

multi linear in $\nabla_{au+bv} w = a\nabla_u w + b\nabla_v w$

The cov. deriv. = ~~rate of~~ change

$$\boxed{\nabla_{\vec{\varepsilon}} w \approx w_{\|}(q \rightsquigarrow p) - w(p) = \delta_{pq} w}$$

Def'n: $\delta_{pq} w$ = small (intrinsic) change in w along $\vec{\varepsilon}$ from p to q.
from tail to tip of $\vec{\varepsilon}$

Thus,
$\Rightarrow$ If $\nabla_v w = 0$ then $\delta_{pq} w = 0 \Rightarrow w$ is P.T.'d along v.

$\Leftarrow$ If w is P.T.'d, $\delta_{pq} w = 0 \Rightarrow \nabla_v w = 0$

geodesic equation
v = velocity of geodesic $\boxed{\nabla_v v = 0}$

Holonomy

Differences b/w M^2 & M^3

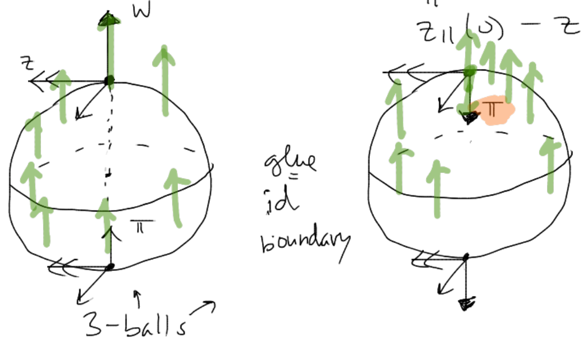
- M^2: holonomy is an angle (scalar) (the PT'd vector $\subset$ Plane of Loop)
- M^3: holonomy is a vector

- M^2: $\mathcal{R}(L)$ – no w! doesn't depend on which vector to transport
- M^3: it does.

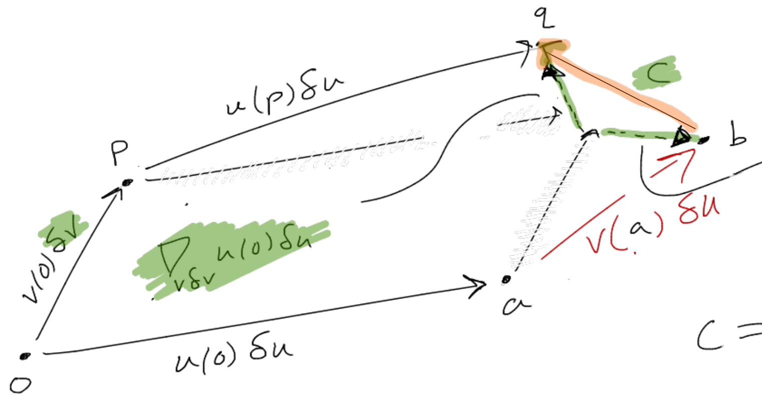
$w_{\parallel}(0) - w(0) = 2\pi$

$z_{\parallel}(0) - z(0) = 0$

ex S^3



M^3



$\nabla_{u\delta u} v(0)\,\delta v$

$$c = \nabla_{v\delta v}(u\,\delta u) - \nabla_{u\delta u}(v\,\delta v)$$

$$= \delta v\,\delta u\,\nabla_v u - \delta v\,\delta u\,\nabla_u v$$

$$= \underbrace{(\nabla_v u - \nabla_u v)}_{[v,u]}\ \delta v\,\delta u$$

$[v,u]$ · the commutator or Lie Bracket

property: Anti-symmetric

$$[v,u] = -[u,v]$$

orientation

Riemann Curvature Tensor

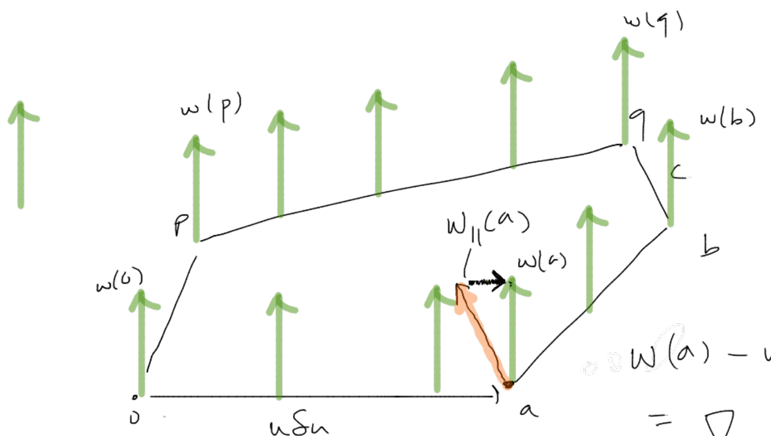
Find change $\delta w_{\parallel}$ when $w_{\parallel}$ is P.T'd around a loop.

① Recall, holonomy is computed via Fiducial V.F.

choose vector field to be w — where w is the vector to be PT'd

② Because then the change you seek: $\delta w_{\parallel} = w_{\parallel}(0 \rightsquigarrow 0) - w(0)$ can be expressed as the covariant derivative of the vector field

negative of



$w(a) - w_{\parallel}(a) = -\delta_{0a} w_{\parallel}$

$= \nabla_{u\delta u}(w(0))$

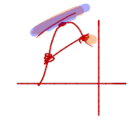
cov deriv of vector Field.

5 segments

$$-\delta w_{11} = -\Big[\underbrace{\delta_{oa} w_{11} + \delta_{ab} w_{11}}_{\text{with orientation}} + \delta_{bq} w_{11} + \underbrace{\delta_{qp} w_{11} + \delta_{po} w_{11}}_{\text{against orient}}\Big]$$

$$= -\Big[-\nabla_{\delta u\, u} w(o) - \nabla_{\delta v\, v} w(a) + \nabla_{c} w(b) + \nabla_{\delta u\, u} w(p) + \nabla_{-v\delta v} w(o)\Big]$$

$$\delta u \delta v [v,u] \qquad -\nabla_u \delta v (w(p))$$



$$=$$

$$\simeq \delta v\big(\underbrace{\nabla_{\delta v}(w(a)) - \nabla_{\delta v} w(0)}_{\text{MVT}}\big) - \delta u\big(\underbrace{\nabla_{\delta u}(w(p)) - \nabla_{\delta u}(w(0))}_{\text{MVT}}\big) - \delta u \delta v \nabla_{[v,u]} w(0)$$

$$\delta v\big(\nabla_u \nabla_{\delta v} w\big) - \delta u\big(\nabla_{v} \nabla_{u} w\big) - \delta u \delta v \nabla_{[v,u]} w$$

$$= \delta v \delta u \big(\overbrace{\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[v,u]}}^{R(u,v)}\big) w$$