MA541 wk 2

Topic: differentiability in $\mathbb{R}^n$, differential, regular surface, tangent space

source: Text: Geometry of Curves & Surfaces, 2nd ed, Do Carmo

Def'n let $f: U \subset \mathbb{R} \longrightarrow \mathbb{R}$ the derivative $f'(x)$ of f @ $x_0 \in U$ is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{w/} \quad x+h \in U$$

- If f has derivatives @ all points in $U \Rightarrow f$ is differentiable on U
- If f has cts derivatives of all orders @ $x_0 \Rightarrow f''(x_0)$, $f'''(x_0)$ exists
- $C^\infty = \{$ infinitely differentiable $\}$

Partial: let $f: U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

the partial deriv. of f w.r.t. x @ $(x_0, y_0) \in U$

$\frac{\partial f}{\partial x_0}(x_0, y_0)$ is the derivative of the function of one variable

$$x \longmapsto f(x, y_0)$$

$\hookrightarrow$ fixed!

2nd Partials:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Sometimes: $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial x \partial y} = f_{xy}$

Important: ① Partials Commute

② Partials obey the chain rule

Partials & Chain Rule

let $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ be differentiable and $f(x,y,z)$ is too

Ex 

f: color det by x,y,z

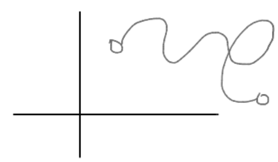
(composite fcn

$f(x(u,v), y(u,v), z(u,v))$ is differentiable &

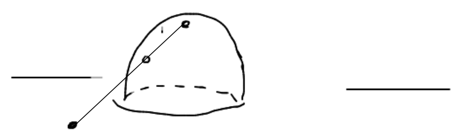
$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

chain rule

Ex $F: (-\epsilon, \epsilon) \subset \mathbb{R} \longrightarrow \mathbb{R}^2$



Ex $F: U \subset \mathbb{R}^3 \to \mathbb{R}^2$

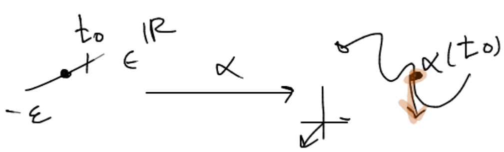


Def'n A tangent vector to a map $\alpha: U \subset \mathbb{R} \longrightarrow \mathbb{R}^m$ @ $t_0 \in U$

$\alpha(t) = (x_1(t), x_2(t), \ldots, x_m(t))$

is the vector in $\mathbb{R}^m$

$$\alpha'(t) = (x_1'(t), x_2'(t), \ldots\ldots \ldots x_m'(t))$$



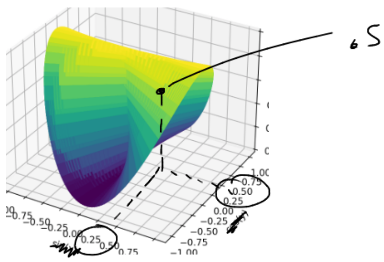
Ex $\alpha: U \subset \mathbb{R} \to \mathbb{R}^3$ by $\alpha(t) = (t^3, t^2, t)$

Ex: Cool surface $F: U \subset \mathbb{R}^2 \to \mathbb{R}^3$

$$F(u,v) = (\cos(u)\cos(v),\ \cos(u)\cdot\sin(v),\ \cos^2(v)) \qquad u,v \in U$$

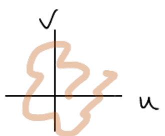
Note: component functions are cts & diff'ble.

$\Rightarrow$ surface they parametrize is 'smooth'



Ex

$$G(u,v) = \left(\left(2 - v\sin\left(\tfrac{u}{2}\right)\right)\cdot\sin(u),\ \left(2 - v\sin\left(\tfrac{u}{2}\right)\right)\cos(u),\ v\cos\left(\tfrac{u}{2}\right)\right)$$

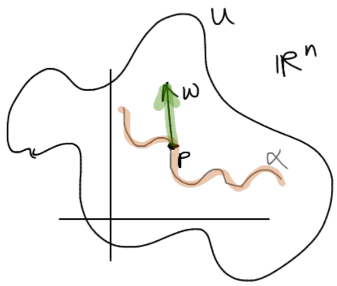


the differential of a differentiable map

Def'n let $F: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a diff'ble map

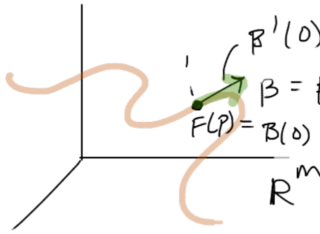
To each $p \in \mathbb{R}^n$ we associate a linear map

$dF_p : \mathbb{R}^n \to \mathbb{R}^m$, called the differential of f @ p



defined: let $w \in \mathbb{R}^n$ & $\alpha : (-\epsilon, \epsilon) \to U$ be diff'ble curve s.t $\alpha(0) = p$, $\alpha'(0) = w$

$$dF_p(w) = \beta'(0)$$



$\beta = F \circ \alpha$ (diff'ble via chain rule)

$F(p) = \beta(0) = F \circ \alpha(0) = F(p)$

Ex: $F: \mathbb{R} \to \mathbb{R}$, $F(x) = \cos(x)$ | $\frac{dF}{dx} = -\sin x$, $\frac{dF}{dx} \approx \frac{\delta F}{\delta x} \approx -\sin x$ | $\delta F \approx -\sin x \, \delta x$

multi. by δx

$dF \approx -\sin x \, dx$

@ a given point $x_0 = \frac{\pi}{3}$

$dF \approx -\sin\frac{\pi}{3} dx$

$dF \approx -\frac{\sqrt{3}}{2} dx$

or

@ $x_0 = \frac{\pi}{4}$

$dF \approx -\frac{\sqrt{2}}{2} dx$

the differential is locally, proportional to dx

See: p.36 Needham

$\left(A = \frac{\partial \hat{s}_1}{\partial u} \approx \frac{\delta \hat{s}_1}{\delta u} \right)$

" proportional"

Prop. the differential is a linear map. (we show this by expressing it as a matrix (in some basis))

Assume $F: U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

(u,v) coords in $\mathbb{R}^2$ $\quad e_1 = (1,0)$, $e_2 = (0,1)$

(x,y,z) coords in $\mathbb{R}^3$ $\quad f_1 = (1,0,0)$, ..., $f_3 (0,0,1)$

$\alpha : U \subset \mathbb{R} \to \mathbb{R}^2$

$$\alpha(t) = (u(t), v(t)) = u(t) \cdot \underbrace{e_1}_{(1,0)} + v(t) \cdot \underbrace{e_2}_{(0,1)}$$

$$\alpha'(t) = (u'(t), v'(t))$$

$$\boxed{\alpha'(0) = u'(0) \cdot e_1 + v'(0) e_2}$$

write in coords $F(u,v) = (x(u,v), y(u,v), z(u,v))$

Image of curve under F $\quad \beta(t) = F \circ \alpha(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))$

$$\frac{d\beta}{dt} = \left(\underbrace{\frac{d}{dt}(x(u(t), v(t))}, \frac{d}{dt} y(u(t), v(t)), \frac{d}{dt} z(u(t), v(t))\right)$$

F(p)

$$= \left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}), \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}), \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t}\right) \in \mathbb{R}^3$$

$$= \left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\right) f_1 + \left(\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}\right) f_2 + \left(\frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t}\right) f_3$$

$$= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix}$$

set $t=0$

$= dF_p(w) \Rightarrow$ thus dF_p is a linear map.

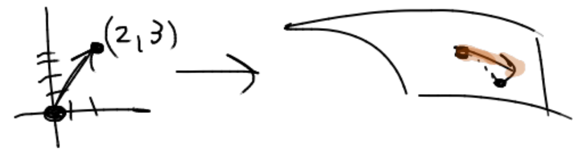
Ex

$$F: \mathbb{R}^2 \to \mathbb{R}^2$$

$$F(x,y) = (x^2 - y^2, 2xy)$$

dF_p @ (x,y) is.

$$= \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$



$(2^2 - 9, 12)$
$(-5, 12)$

and

$$dF_{(1,1)} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$ apply to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$dF_{(1,1)}(2,3) = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$$

this is an example of dF being a linear map.