

Differential Geometry: Homework 1 Solutions

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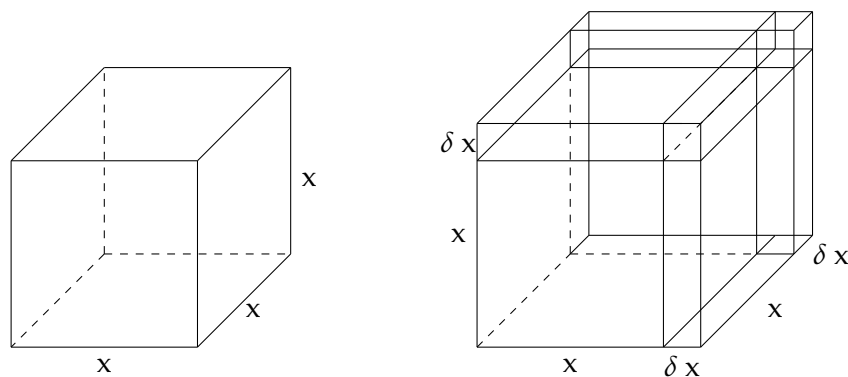
1. (This is a model of how an “ultimate equality” - see Prologue - becomes an equality.) Sketch a cube of side x , hence of volume $V = x^3$. In the same picture, keeping one vertex fixed, sketch a slightly larger cube with side $x + \delta x$. If δV is the resulting increase in volume, use your sketch to deduce that as δx vanishes,

$$\delta V \asymp 3x^2\delta x \Rightarrow \frac{dV}{dx} \asymp \frac{\delta V}{\delta x} \asymp 3x^2 \Rightarrow \frac{dV}{dx} \asymp 3x^3$$

But *the quantities in the final ultimate equality are independent of δx* , so they are equal:

$$(x^3)' = \frac{dV}{dx} = 3x^2$$

Solution:



As we easily can see in the above images, as δx vanishes, we will be left with only the three sides of the cube that δV was covering left as constant. These three sides have a combined surface area of $3x^2$.

this can also be shown algebraically by solving for δv in $(v + \delta v) - v$:

$$(v + \delta v) - v \asymp (x + \delta x)^3 - x^3$$

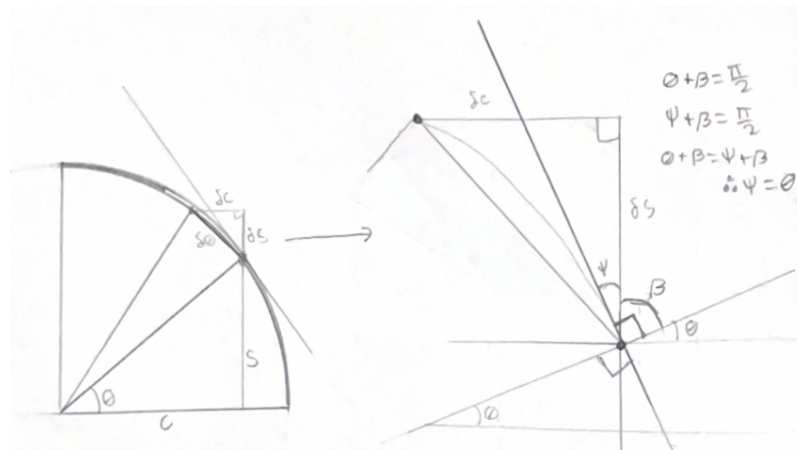
If we remove all higher order terms of δ from our expansion, ($\delta^2 x$ and higher, as these are essentially zero) we have

$$v + \delta v - v \asymp (x + \delta x)^3 - x^3 \Rightarrow \delta v \asymp x^3 + 3x^2\delta x - x^3 \Rightarrow \delta v \asymp 3x^2\delta x$$

$$\Rightarrow \frac{\delta v}{\delta x} \asymp 3x^2 \Rightarrow \frac{dv}{dx} = 3x^2$$

which proves our claim that as δx vanishes, we are left with only the surface area of the vanishing δv .

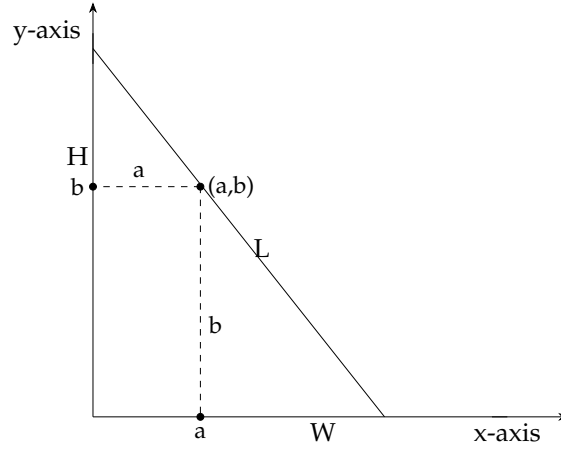
- Let $c = \cos(\theta)$ and $s = \sin(\theta)$. In the first quadrant of $\mathbb{R}^2$, draw a point $p = (c, s)$ on the unit circle. Now let p rotate by a small (ultimately vanishing) angle $\delta\theta$. with one vertex at p , draw the small triangle whose sides are δc and δs . By emulating the Newtonian geometrical argument in the prologue, "instantly and simultaneously" deduce that $\frac{\delta s}{\delta\theta} = c$ and $\frac{\delta c}{\delta\theta} = -s$.



As we can see on the above image, as $\delta\theta$ becomes infinitely small, the triangle which is overlapping the circle will become the triangle along the tangent line that intersects point $p = (c, s)$. The triangle on this tangent is a similar triangle to the original, however it is now rotated. This is proven from the relationships: $\theta + \beta = \frac{\pi}{2}$, $\psi + \beta = \frac{\pi}{2}$ thus, $\theta = \psi$ as shown .

Hence, we have: $\frac{ds}{d\theta} \asymp \frac{\delta s}{\delta\theta} \asymp \frac{c}{1}$ and $\frac{dc}{d\theta} \asymp \frac{\delta c}{\delta\theta} \asymp -\frac{s}{1}$, which is negative because of the direction.

3. Let L be a general line through the point (a, b) in the first quadrant of $\mathbb{R}^2$, and let A be the area of the triangle bounded by the x-axis, the y-axis, and L .
- (i) Use ordinary calculus to find the position of L that minimizes A , and show that $A_{min} = 2ab$.



our triangles area is $A = \frac{1}{2}HW$. We want to find the position of L that minimized this area while traveling through the point (a, b) . Using similar triangles, we have

$$\frac{H}{W} = \frac{b}{W-a} \Rightarrow H = \frac{Wb}{W-a}$$

Substituting this value into our area expression for H , we can differentiate our expression and set it equal to zero.

$$A = \frac{1}{2} \frac{W^2 b}{W-a} \Rightarrow \frac{dA}{dW} = 0 = \frac{1}{2} \frac{2Wb}{W-a} - \frac{1}{2} \frac{W^2 b}{(W-a)^2}$$

cleaning up like like terms gives

$$0 = 1 - \frac{1}{2} \frac{W}{W - a} \Rightarrow W = 2a$$

Substituting $W = 2a$ back into our similar triangles expression gives:

$$H = \frac{2ab}{2a - a} \Rightarrow H = 2b$$

Thus, if we plug H and W back into our expression, we will have the minimum area of:

$$A_{min} = \frac{1}{2}(2b)(2a) \Rightarrow A_{min} = 2ab$$

(ii) Use Newtonian reasoning to solve the problem instantly, without calculation! (Hints: Let δA be the change in the area resulting from a small (ultimately vanishing) rotation $\delta\theta$ of L . By drawing δA in the form of two triangles, and observing that each triangle is ultimately equal to a sector of a circle, write down an ultimate equality for δA in terms of $\delta\theta$. Now set $\delta A = 0$.)

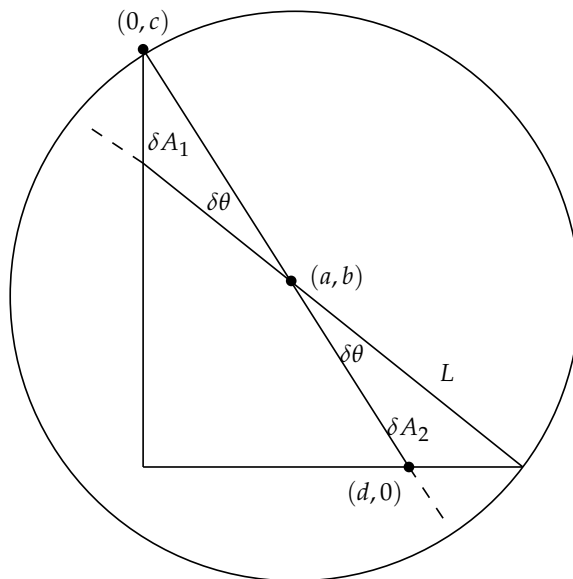


Figure 1: Small rotation $\delta\theta$ of line L producing regions δA_1 and δA_2 .

From this image, we have rotated L clockwise by some amount $\delta\theta$. giving us $\delta A = \delta A_1 - \delta A_2$. For δA_1 and δA_2 , we can plug in the sector of a circle areas, which is $\delta A_1 = \frac{1}{2}r_1^2\delta\theta$ and $\delta A_2 = \frac{1}{2}r_2^2\delta\theta$. Using the distance formula, we can see

$$r_1^2 = \left(\sqrt{(0-a)^2 + (c-b)^2} \right)^2 = a^2 + (c-b)^2$$

$$r_2^2 = \left(\sqrt{(a-d)^2 + (b-0)^2} \right)^2 = (a-d)^2 + b^2$$

plugging everything in for δA gives us

$$\delta A = \frac{1}{2}(a^2 + (c-b)^2)\delta\theta - \frac{1}{2}((a-d)^2 + b^2)\delta\theta$$

dividing out the $\delta\theta$, setting equal to zero, and dividing out the $\frac{1}{2}$ gives

$$\Rightarrow \frac{\delta A}{\delta\theta} = a^2 + (c-b)^2 - (a-d)^2 + b^2 = 0$$

$$\Rightarrow a^2 + b^2 + c^2 - 2bc - a^2 - b^2 - d^2 + 2ad = 0$$

$$\Rightarrow c^2 - 2bc - d^2 + 2ad = 0$$

$$\Rightarrow c(c-2b) - d(d-2a) = 0$$

this final statement is true when $c = 2b$ and $d = 2a$ as we should expect and as shown in the first problem. This may be written in ultimate equivalence form as

$$\frac{dA}{d\theta} \asymp \frac{\delta A}{\delta\theta} \asymp 0 \text{ precisely when } c = 2b \text{ and } d = 2a$$

which again proves that $A_{min} = \frac{1}{2}(2a)(2b) = 2ab$

4. Prove the transitive property of ultimate equality.

We want to prove that if $A \asymp B$ and $B \asymp C$ then $A \asymp C$.

$$A \asymp B \asymp B \asymp C \Rightarrow \lim_{\epsilon \rightarrow 0} \frac{A}{B} \asymp \lim_{\epsilon \rightarrow 0} \frac{B}{C} \Rightarrow \lim_{\epsilon \rightarrow 0} \frac{\frac{A}{B}}{\frac{B}{C}} = 1$$

$$\Rightarrow \lim_{\epsilon \rightarrow 0} \frac{A}{B} \cdot \frac{C}{B} = 1 \quad \Rightarrow \quad \lim_{\epsilon \rightarrow 0} \frac{B}{A} \cdot \frac{C}{B} = 1 \quad \Rightarrow \quad \lim_{\epsilon \rightarrow 0} \frac{C}{A} = 1$$

Which Proves $C \asymp A$ and thus, by our proof we have done in class, $A \asymp C$.