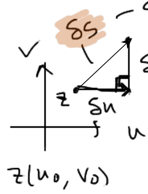
MA541 wk 3 Mon.

4.3 – 4.5 Conformal Maps.

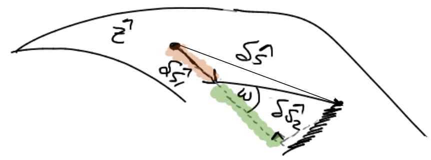
Exercise.

First Fundamental Form:

δs – change in dist under small chang in coords



$\psi : \mathbb{C} \to \Sigma$



$$\delta s^2 = \delta u^2 + \delta v^2$$

vector!

$$A \equiv \frac{\delta \hat{s}_1}{\delta u} \approx \frac{\partial \hat{s}_1}{\partial u} \Rightarrow \delta \hat{s}_1 = A \cdot \delta u$$

constant (depending on z) capturing scale factor in horiz. (u) direction

$$B \equiv \frac{\delta \hat{s}_2}{\delta v} \approx \frac{\partial \hat{s}_2}{\partial v} \Rightarrow \delta \hat{s}_2 = B \delta v$$

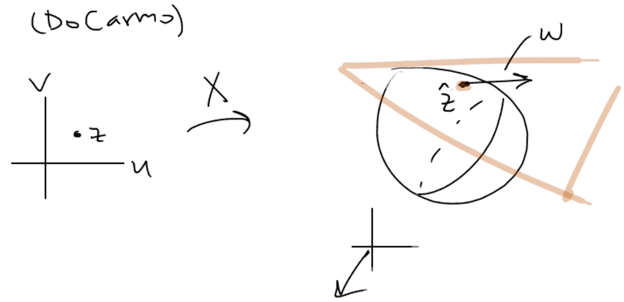
$$\delta \hat{s}^2 \approx (\delta \hat{s}_1 + \delta \hat{s}_2 \cos(\omega))^2 + (\delta \hat{s}_2 \sin \omega)^2$$

$$\approx \delta \hat{s}_1^2 + 2 \delta \hat{s}_1 \delta \hat{s}_2 \cos(\omega) + \delta \hat{s}_2^2 \cos^2(\omega) + \delta \hat{s}_2^2 \sin^2 \omega$$

$$\approx \underbrace{\delta \hat{s}_1^2}_{A^2 \delta u^2} + 2 \underbrace{\delta \hat{s}_1}_{A \delta u} \underbrace{\delta \hat{s}_2}_{B \delta v} \cos \omega + \underbrace{\delta \hat{s}_2^2}_{B^2 \delta v^2}$$

$$ds^2 = A^2 du^2 + 2AB\cos(\omega)\, du\, dv + B^2 dv^2$$

Classical Derivation of F.F.F.
(DoCarmo)



the inner product in $\mathbb{R}^3$ (things that gives distances in $\mathbb{R}^3$) induces inner prod. on $T_{\hat{z}}(S)$ ← tan. plane @ $\hat{z}$.

Every inner-product ⟷ bilinear form (symmetric bilinear)
For $w \in T_p(S)$.
So we get $I_p(w) = \langle w, w \rangle = |w|^2$.

so we see that the first fund. form is is an expression of how the inner product (distance) of the ambient space is inherited in the surface

Basis for $T_p(S)$: $\{X_u, X_v\}$ ∵ $X_u = A$, $X_v = B$ (Needham)

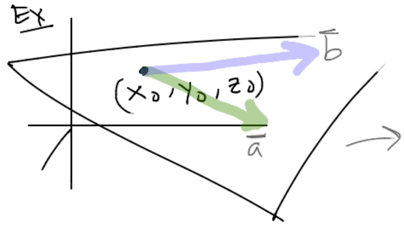
It's straight forward to see $w = \alpha'(0)$ ← initial tan. vector of curve in S

$\alpha'(0) = X_u \cdot u' + X_v v'$ expressing α' in Basis !

$$I_p(w) = I_p(\alpha'(0)) = I_p(X_u \cdot u' + X_v \cdot v')$$
$$= \langle X_u u' + X_v v', X_u u' + X_v v' \rangle$$
$$= \langle X_u, X_u \rangle (u')^2 + 2\langle X_u, X_v \rangle u'v' + \langle X_v, X_v \rangle (v')^2$$
$$= E(u')^2 + 2Fu'v' + G(v')^2$$

$E = \sqrt{A}$
$G = \sqrt{B}$

Ex



$\mathbb{R}^3$

Plane thru $(x_0, y_0, z_0) = p$

Basis: $\bar{a} = (a_1, a_2, a_3)$

$\bar{b} = (b_1, b_2, b_3)$, $\bar{a}, \bar{b}$ orthonormal

parametrize this plane

$$X(u,v) = p + u\bar{a} + v\bar{b}$$

observe the F.F.F. is just the Pythagorean thm:

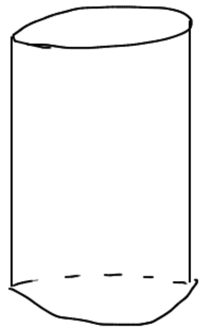
$$E = 1, \quad F = 0, \quad G = 1$$

let $w = p + u\bar{a} + v\bar{b}$

$$I_p(w) = I_p(p + u\bar{a} + v\bar{b}) = \langle p + u\bar{a} + v\bar{b}, p + u\bar{a} + v\bar{b} \rangle$$

exercise.

Ex 2



$x(u,v) = (\cos u, \sin u, v)$

$u \in (0, 2\pi)$

$v \in \mathbb{R}$

$x_u = (-\sin u, \cos u, 0)$

$x_v = (0, 0, 1)$

$E = \langle x_u, x_u \rangle = 1$

$G = \langle x_v, x_v \rangle = 1$

$F = \langle x_u, x_v \rangle = 0$

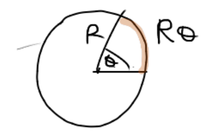
$\Rightarrow ds^2 = 1\,du^2 + 0\,du\,dv + 1\,dv^2$

$ds^2 = du^2 + dv^2$

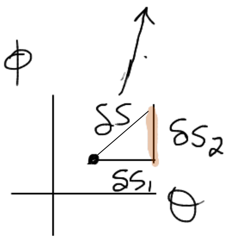
very cool! this is the same as the flat plane …. the cylinder and the plane have the same intrinsic geometry

4.4 Spherical Metric w/ lat, lon coords

coordinatize sphere (θ, ϕ) — latitude
radius R — longitude



$$c(r) = 2\pi R \sin\phi$$
$$= 2\pi R \sin\left(\frac{r}{R}\right)$$



What does a small change in θ, δs_1

(1) induce on sphere

$d\hat{s}_1$ vs. ds_1

$$\boxed{\frac{d\hat{s}_1}{d\theta} = R}$$

(2) $\boxed{d\hat{s}_2 = R\sin\phi \, d\phi}$ $\qquad \delta\hat{s}_2 \approx R\sin\phi \, \delta\phi$

Finally
$$d\hat{s}^2 = d\hat{s}_1^2 + d\hat{s}_2^2$$
$$= R^2 d\theta^2 + R^2 \sin^2\phi \, d\phi^2 = R^2\left(d\theta^2 + \sin^2\phi \, d\phi^2\right)$$

Curvature Metric Function:

For a general surface

w/ A, B as before $A=\sqrt{E}$, $B=\sqrt{G}$

$$K = \frac{-1}{AB}\left(\frac{\partial_v(\partial_v A)}{B} + \frac{\partial_u(\partial_u B)}{A}\right)$$

λ

λ
$\shortparallel$

when $A=B$ the deformations (scale factors) are equal in both directions
$\Rightarrow$ Conformal
$\Rightarrow$ preserve angles
$\Rightarrow$ this formula becomes simpler!

$$K = -\nabla\frac{(\ln(\lambda))}{\lambda^2}$$

Compute K for spherical / lat-lon metric

$$K = \frac{-1}{AB}\left(\frac{\partial_v(\partial_v A)}{B} + \frac{\partial_u(\partial_u B)}{A}\right)$$

Metric $\quad u = \theta \;,\; v = \phi \;,\; A = R\sin\phi$

$$B = R$$

$$= \frac{-1}{R^2 \sin\phi}\left(\partial_v(\ldots \quad \ldots\right) = \frac{1}{R^2}$$