Outline (A) Alex's Question: Linear Approx.
(B) 4.3 Metric of General Surface
(C) 4.4 Metric Curvature formula
~~4.8 Stereographic Projection~~
(D) 4.5 Conformal Maps
~~4.7 Conformal St. Maps of S^2~~
(E) #9, P. 85

(B) (4.3) F.I.F.F. $\quad \psi : \mathbb{C} \to F \quad \psi(z) = \hat{z}$

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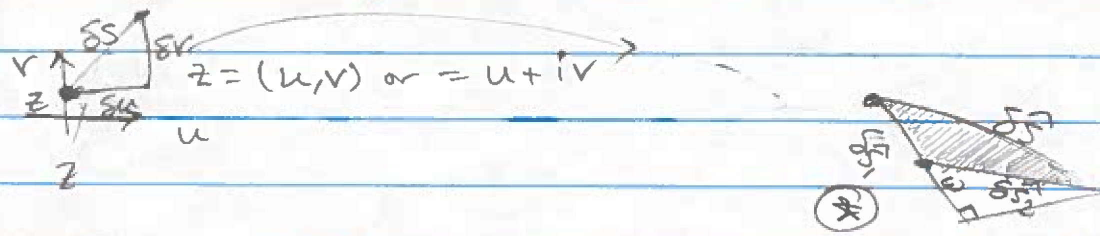

$z = (u, v)$ or $= u + iv$

$$ds^2 = du^2 + dv^2$$
or $ds_1^2 + ds_2^2$

FACT: small horizontal movement δu ultimately produces a proportional movement $\delta \hat{s}_1$

$\delta \hat{s}_1 \asymp A\,\delta u$ along v-curve

$$\frac{\partial \hat{s}_1}{\partial u} \equiv A \asymp \frac{\delta \hat{s}_1}{\delta u}$$

$$\frac{\partial \hat{s}_2}{\partial v} = B \asymp \frac{\delta \hat{s}_2}{\delta v}$$

means that since ultimately on the surface we see this (*) rt Δ and P.T. $\Rightarrow$

$$\delta \hat{s}^2 \asymp (\delta \hat{s}_1 + \delta \hat{s}_2 \cos\omega)^2 + (\delta \hat{s}_2 \cdot \sin\omega)^2$$
$$\asymp (A\delta u + B\delta v \cos\omega)^2 + (B\delta v \sin\omega)^2$$
$$\asymp A^2\delta u^2 + B^2\delta v^2 + 2AB\cos\omega\,\delta u\,\delta v$$

F.I.F.F. $\quad ds^2 = A^2 du^2 + \underbrace{2AB\cos\omega}_{F}\,du\,dv + B^2 dv^2$

How surface inherits metric from $\mathbb{R}^3$.

see DoCarmo:

(B)

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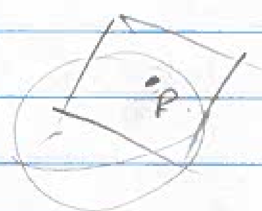
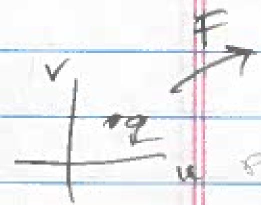
Tangent Plane:





$p: F(u,v) = (F_1(u,v), F_2(u,v), F_3(u,v))$

$dF_q : \mathbb{R}^2 \to T$

quadratic form on $T_p(S)$ defined by

$$I_p(w) = \langle w, w \rangle_p = |w|^2$$

Express FFF in basis $\{x_u, x_v\}$ associated to param $x(u,v)$ @ p.

Tang. vector in $T_p(S)$ is w, and is tang. vector to some curve $\alpha(t)$

$$w = \alpha'(0);$$

$\alpha'(0)$ in the basis: $\alpha'(0) = x_u u' + x_v v'$

so

$$I_p(w) = I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle = \langle x_u \overset{A}{u'} + x_v \overset{B}{v'}, x_u u' + x_v v' \rangle$$

$$= \langle x_u, x_u \rangle_p (u')^2 + 2 \langle x_u, x_v \rangle u'v' + \langle x_v, x_v \rangle_p (v')^2$$

$$= E(u')^2 + 2Fu'v' + G(v')^2$$

coeffs

B

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Ex

$w_2 = (b_1, b_2, b_3)$

$w_1 = (a_1, a_2, a_3)$

(x_0, y_0, z_0)

w/ $w_1 \perp w_2$ by assumption

orthonormal

$$x(u,v) = P_0 + u w_1 + v w_2 \qquad u, v \in \mathbb{R}^2$$

Note. $x_u = w_1$

$x_v = w_2$

orthogonality $\Rightarrow$ $F = 0$.

Here, the F.F.F. is Pyth. thm $\quad E = 1,\ F = 0,\ G = 1$

let $w = P_0 + c w_1 + d w_2$

$$I_p(w) = I_p(P_0 + c w_1 + d w_2)$$
$$= E(u')^2 + F u' v' + G (v')^2$$

$$\left.\begin{aligned} E = \langle x_u, x_u \rangle &= \langle w_1, w_1 \rangle = 1 \\ \langle x_v, x_v \rangle &= 1 \\ \langle x_u, x_v \rangle &= 0 \end{aligned}\right\}$$

P. 96

Ex 2

$$x(u,v) = (\cos u, \sin u, v)$$
$$U = \{(u,v) \mid 0 < u < 2\pi,\ -\infty < v < \infty\}$$

Compute F.F.F. $\quad x_u = (-\sin u, \cos u, 0)$

$x_v = (0, 0, 1)$

$$E = \langle x_u, x_u \rangle = \sin^2 u + \cos^2 u = 1$$

$G = 1 = \langle x_v, x_v \rangle$

$F = 0$

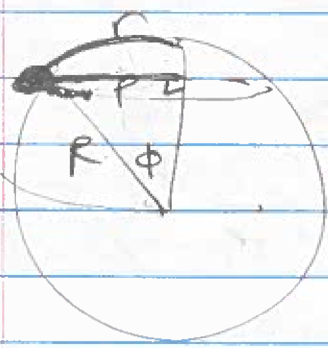
C

(I)

Ex Spherical Metric w/ lat. lon coords

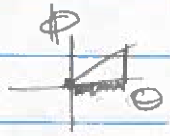
(θ, ϕ)

$$d\hat{s}^2 = R^2\left[\sin^2\phi \, d\theta^2 + d\phi^2\right] = A^2 d\theta + B^2 d\phi^2$$



$$C(r) = 2\pi R \sin(r/R)$$

$\sin\phi = \rho/R$



① $d\hat{s}^2 = d\hat{s}_1^2 + ds_2^2$

$=$

$\rho = R\sin\phi$

$\delta\hat{s}_1 \simeq R\sin\phi\,\delta\theta$ horiz. movement in θ — radius dark arc

$\delta\hat{s}_2 \simeq R\delta\phi$ — use radius

$\frac{ds_1}{d\theta} = R\sin\theta$

$\frac{ds_2}{d\phi} = R$

$\hat{s}_1 = -R\cos\theta$

$s_2 = R\phi$

② $\delta s^2 \simeq R^2\sin^2\phi\,\delta\theta^2 + \delta\phi^2$

(II)

In this $u = \theta$, $v = \phi$, $A = R\sin\phi$

$B = R$

$(R\cos\theta, 0) : \hat{s}_1$

$(0, R\phi) : \hat{s}_2$

$$\Rightarrow K = \frac{-1}{AB}\cdot\left(\partial_v \frac{(\partial_v A)}{B} + \partial_u \frac{(\partial_u B)}{A}\right)$$

$(R\sin\theta, 0) : \hat{s}_{1u}$

$(0, R) : \hat{s}_{2v}$

$$= \frac{1}{-R^2\sin\phi}\left(\partial_v \frac{(R\cos\phi)}{B} + \partial_u \frac{(0)}{A}\right) = \frac{1}{-R^2\sin\phi}\cdot\frac{-R\sin\phi}{R}$$

$$= \frac{1}{R^2}$$

AREA ELEMENT

$\Rightarrow dA = AB\sin\omega\, du\, dv$

$= A\delta u\, B\delta v \cdot \sin(\omega)$

Conformal

$\leftarrow$

$= (A^2B^2)\sin^2(\omega)\, du^2 dv^2$

$= (A^2B^2 - A^2B^2\cos^2\omega)\, du^2 dv^2$

show

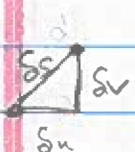
$$dA = \sqrt{(AB)^2 - F^2}\, du\, dv \iff dA^2 = ((AB)^2 - F^2)\, du^2 dv^2$$

1)

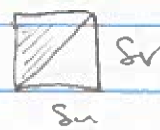
$\delta A \approx A\delta u \cdot B\delta v \sin\omega = AB\sin\omega\, \delta u \delta v$

1/2

goal:

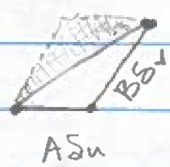


and



$\delta\alpha = \delta s^2$

$\downarrow$



area $A\delta u \cdot B\delta v \cdot \sin(\omega)$

in orthonormal coords: $dA = AB\, du\, dv$

(E)

(4.16)

Conformal Curvature Formula $K = -\frac{\nabla^2 \ln \Lambda}{\Lambda^2}$

apply it to metric on sphere under stereographic proj.:

$$d\hat{s} = \frac{2}{1+(r/R)^2}\, ds \qquad (4.22)$$

(4.22) ⇒ the constant we seek $\Lambda = \frac{2}{1+(r/R)^2} = \frac{2R^2}{R^2+x^2+y^2}$

Begin w/ derivs! (prep for ∇^2)

$$\partial_x \Lambda = \frac{-4R^2 x}{(R^2+x^2+y^2)^2}$$

$$\partial_y \Lambda = \frac{-4R^2 y}{(R^2+x^2+y^2)^2}$$

This is *almost* $\Lambda^2 = \frac{4R^2}{(R^2+x^2+y^2)^2}$ so

1 $\partial_x \Lambda = -\frac{\Lambda^2 x}{R^2}$ & $\partial_y \Lambda = -\frac{\Lambda^2 y}{R^2}$ now:

$\partial_x(\ln\Lambda) = \frac{\partial_x \Lambda}{\Lambda} = \frac{-\frac{\Lambda^2 x}{R^2}}{\Lambda} = \frac{-\Lambda x}{R^2}$, $\boxed{\partial_y(\ln\Lambda) = \frac{-\Lambda y}{R^2}}$

2 $\partial_x^2(\ln\Lambda) = -\frac{1}{R^2}[\partial \Lambda \cdot x] = -\frac{1}{R^2}[\partial \Lambda x] = -\frac{1}{R^2}[\partial_x \Lambda \cdot x + \Lambda]$

$= -\frac{1}{R^2}\left[\frac{-\Lambda^2 x^2}{R^2} + \Lambda\right]$

& same for:

$$\partial_y^2(\ln\Lambda) = -\frac{1}{R^2}\left[\frac{-\Lambda^2 y^2}{R^2} + \Lambda\right]$$

$$K = \frac{-\nabla(\ln \Lambda)}{\Lambda^2} = -\left[\frac{\partial_x^2(\ln\Lambda) + \partial_y^2(\ln\Lambda)}{\Lambda^2}\right]$$

$$= -\left[\frac{\frac{-1}{R^2}\left[\frac{-\Lambda^2 x^2}{R^2}+\Lambda\right] - \frac{1}{R^2}\left[\frac{-\Lambda^2 y^2}{R^2}+\Lambda\right]}{\Lambda^2}\right]$$

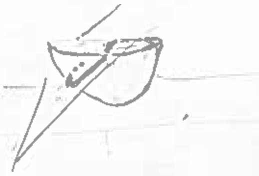
$$= \frac{1}{R^2}\left[\frac{-\Lambda^2\left(\frac{r}{R}\right)^2 + 2\Lambda}{\Lambda^2}\right] = \frac{-1}{R^2}\left[-\left(\frac{r}{R}\right)^2 + \frac{2}{\Lambda}\right] =$$

and since $\frac{2}{1+\left(\frac{r}{R}\right)^2} = \Lambda$, $\xrightarrow[\times 2]{\text{invert}} \frac{2}{\Lambda} = 1 + \left(\frac{r}{R}\right)^2$

$$K = \frac{-1}{R^2}\left[-\left(\frac{r}{R}\right)^2 + 1 + \left(\frac{r}{R}\right)^2\right]$$

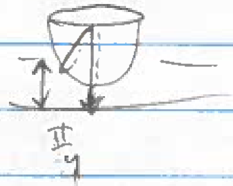
$$K = \frac{1}{R^2}$$

Conformal Maps



(Central) Projective Map of Sphere:
preserves lines (not angles)

2/2

Ex  lines that intersect @ $\frac{\pi}{4}$ $\mapsto$ parallel

$$\boxed{d\hat{s} = \Lambda(z)\,ds}$$

conformal coords:
orthogonal & conformal: preserve circles & angles

$$ds^2 = \Lambda^2(du^2 + dv^2)$$
since $A^2 = B^2 = \Lambda^2$
and $F = 0$

Conformal Curvature $\quad K = \dfrac{-\nabla^2 \ln \Lambda}{\Lambda^2}$