MA541 - Wk 3 - W

HW #10 Note: goal:

$$d\hat{s}^2 = du^2 + dv^2 = \frac{R^2 - y^2}{R^2}\,dx^2 + \frac{R^2}{R^2 - y^2}\,dy^2$$

Today

Ex # 9, p.85 (Monday, Notes p, 6-7)

Exercises #1 (Coord. indepence of K)

Exercise

Conformal Curvature Formula: $K = -\frac{\nabla^2 \ln(\Lambda)}{\Lambda^2}$ $\quad \nabla^2 = \partial_x^2 + \partial_y^2$

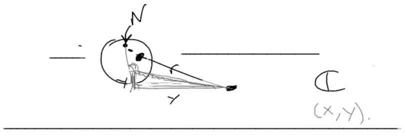
(Orthogonal Metrics:

$d\hat{s}^2 = A\,dx^2 + B\,dy^2$

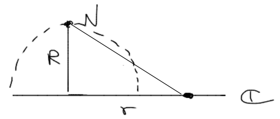
$A = B = \Lambda$ above

Conformal metric $\quad d\hat{s}^2 = \Lambda\, ds^2$, $\quad ds^2 = dx^2 + dy^2$

Stereographic Projection:



(cross-section)



start

$$d\hat{s}^2 = \frac{2}{1+(r/R)}\,ds \qquad (4.22)$$

$$\text{So}\quad \Lambda = \frac{2}{1+(r/R)^2} = \frac{2}{\frac{R^2}{R^2}+\frac{r^2}{R^2}} = \frac{2}{\frac{R^2+r^2}{R^2}} = \left\lceil \frac{2R^2}{R^2+x^2+y^2} \right\rfloor$$

This is "almost"

$$\partial_x\Lambda = \frac{-2R^2\cdot 2x}{(R^2+x^2+y^2)^2} = \frac{-4R^2x}{(R^2+x^2+y^2)^2} = \frac{-\Lambda^2 x}{R^2}$$

$$\partial_y\Lambda = \frac{-4R^2y}{(R^2+x^2+y^2)^2} = -\frac{\Lambda^2 y}{R^2}$$

$$\text{b/c}\quad \Lambda^2 = \frac{4R^4}{(R^2+x^2+y^2)^2}$$

To prepare for $\nabla^2(\ln\Lambda)$

$$\partial_x^2(\ln\Lambda) = \partial_x(\partial_x(\ln\Lambda)) = \partial_x\left(\frac{\partial_x\Lambda}{\Lambda}\right) = \partial_x\left(\frac{\left[\frac{-\Lambda^2 x}{R^2}\right]}{\left(\frac{\Lambda}{1}\right)}\right) = \frac{-1}{R^2}\partial_x(\Lambda\cdot x)$$

$$= -\frac{1}{R^2}\left[\partial_x\Lambda\cdot x + \Lambda\cdot 1\right] = -\frac{1}{R^2}\left[-\frac{\Lambda^2x^2}{R^2} + \Lambda\right]$$

$$= \partial_y^2(\ln\Lambda) = -\frac{1}{R^2}\left[-\frac{\Lambda^2y^2}{R^2} + \Lambda\right] \longleftarrow$$

$$K = -\frac{\nabla^2\ln(\Lambda)}{\Lambda^2} = -\left[\frac{\partial_x^2(\ln\Lambda) + \partial_y^2(\ln\Lambda)}{\Lambda^2}\right] = -\left[\frac{-\frac{1}{R^2}\left[\frac{-\Lambda^2x^2}{R^2}+\Lambda\right] - \frac{1}{R^2}\left[\frac{-\Lambda^2y^2}{R^2}+\Lambda\right]}{\Lambda^2}\right]$$

$$= \frac{1}{R^2}\left[\frac{\frac{-\Lambda^2\overset{=r^2}{(x^2+y^2)}}{R^2} + 2\Lambda}{\Lambda^2}\right] = \frac{1}{R^2}\left[-\frac{r^2}{R^2} + \frac{2}{\Lambda}\right] = \frac{1}{R^2}\left[\frac{-r^2}{R^2} + 1 + \left(\frac{r}{R}\right)^2\right]$$

$$= \frac{1}{R^2} \quad 🙂$$

$$\Lambda = \frac{2}{1+\left(\frac{r}{R}\right)^2} \Rightarrow \text{invert \& } \times 2 \Rightarrow 2\frac{1}{\Lambda} = \frac{1+\left(\frac{r}{R}\right)^2}{\cancel{2}}\cdot\cancel{2} = \frac{2}{\Lambda}$$

Note: these metrics are infinitessimal metrics

often metrics are defined as

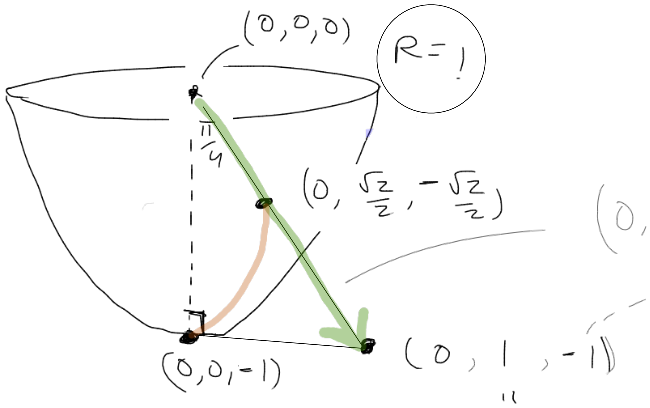
$$\rho : X \times X \longrightarrow \mathbb{R}$$

$$\rho(x,y) = f(x,y) = \begin{cases} \sqrt{\Sigma (x_i - y_i)^2} & \text{eucl.} \\ \dfrac{\sqrt{\Sigma (x_i - y_i)^2}}{y} & \text{hype} \end{cases}$$

Ex In $\mathbb{R}^2$,

"metric" $\equiv \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ works to give dist. for all points

Compare with the central projection metric:



$(0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})t \quad t \in (0, \infty)$

$-\frac{\sqrt{2}}{2}t = -1$

$\Rightarrow t = -1\left(\frac{2}{-\sqrt{2}}\right) = \sqrt{2}$

$1 = \frac{\sqrt{2}}{2}t\Big|_{t=\sqrt{2}}$

Let α = arc from $(0,0,-1)$ following geodesic up to $(0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$

Metric of Sphere under St. Proj

$$d\hat{s}^2 = \frac{1}{[1+(r/R)^2]}\left[\frac{dr^2}{[1+(r/R)^2]} + r^2 d\theta^2\right]$$

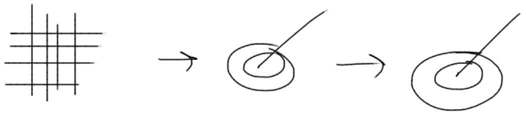
In α, $d\theta = 0$

$$d\hat{s}^2 = \frac{dr^2}{((1+(r/R)^2)^2} \overset{\sqrt{}}{\Rightarrow} ds = \frac{1}{1+r^2}dr \quad \boxed{R=1}$$

$$\text{length} = \int_0^1 ds = \int_0^1 \frac{1}{1+r^2}dr = \tan^{-1}(r)\Big|_0^1 = \underset{=\pi/4}{\overset{''\ \pi/4}{\tan^{-1}(1)}} - \overset{''\ 0}{\tan^{-1}(0)}$$

More Coordinate Independance of K

Start w/ $ds^2 = dx^2 + dy^2$.



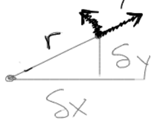
polar coords:

$u = r$

$v = \theta$

$x = r\cos\theta$

$y = r\sin\theta$



vary r, new length

$\delta s_1 = \delta r$ ✓

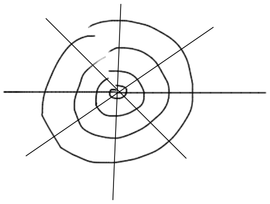
$\delta s_2 = r\,\delta\theta$ ✓

$\delta s^2 = \delta s_1^2 + \delta s_2^2$

$= \delta r^2 + r^2 \delta\theta$

(true, but why? variation in r is ⊥ to variation θ

$$\circledast \quad ds^2 = dr^2 + r^2 d\theta^2$$



Next

choose: $r = u^2$ $\quad x = r\cos\theta$

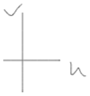
$\theta \overset{\checkmark}{=} v$ $\quad y = r\sin\theta$

$d\hat{s}_1 = dr$

$\hat{s}_1 = r$

$\hat{s}_2 = r\theta$

$d\hat{s}_2 = r\,d\theta$



$dr = 2u\,du$

$d\theta = dv$

↓ algebra

$r\,d\theta = u^2 dv$

$\circledast$

$$d\hat{s}^2 = d\hat{s}_1^2 + d\hat{s}_2^2$$
$$= (2u\,du)^2 + (u^2 dv)^2$$

$$\boxed{d\hat{s}^2 = 4u^2 du^2 + u^4 dv^2}$$

show $K = 0$

$$K = -\frac{1}{AB}\left(\overset{= 0}{\partial_v\frac{(\partial_v(A))}{B}} + \overset{= 0}{\partial_u\frac{(\partial_u(B))}{A}}\right) = 0$$

$$A = \frac{\partial \hat{s}_1}{\partial u} = \frac{\partial r}{\partial u} = 2u \quad , \quad B = \frac{\partial \hat{s}_2}{\partial v} = \frac{\partial(r\theta)}{\partial v} = \frac{\partial(r\,v)}{\partial v} = r$$

u indep of v

$$\frac{\partial_v(A)}{B} = \frac{\partial_v(2u)}{B} = 0 \quad , \quad \partial_v\left(\frac{\partial_v A}{B}\right) = \frac{\partial(0)}{B} = 0$$

$$\frac{\partial_u B}{A} = \underset{\text{exercise}}{} = 1 \quad , \quad \partial_u\left(\frac{\partial B}{A}\right) = \partial_u(1) = 0$$