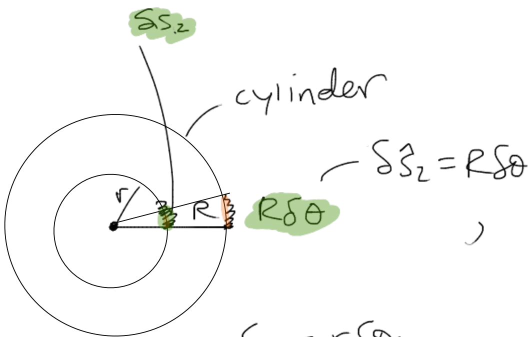
#10/

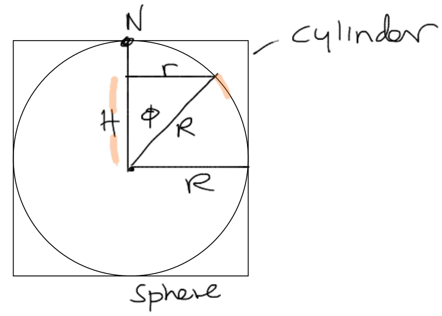
Top view: Horizontal slice



$\delta s_2 = r\delta\theta$

$ds_2 \approx \delta s_2$

Side view: Vertical slice



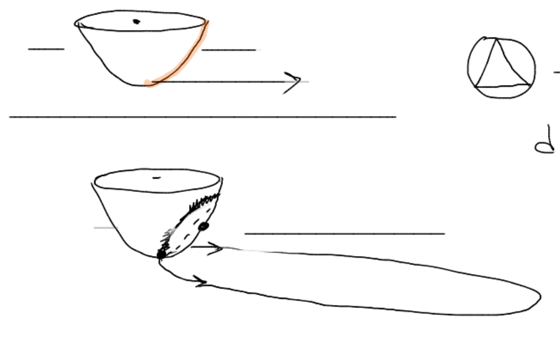
$ds_1 \approx \delta s_1 \approx R\delta\phi$

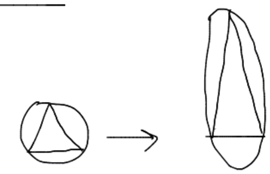
Pseudosphere + Hyperbolic Geometry

Recall, central projection:

preserves lines

distorts angles

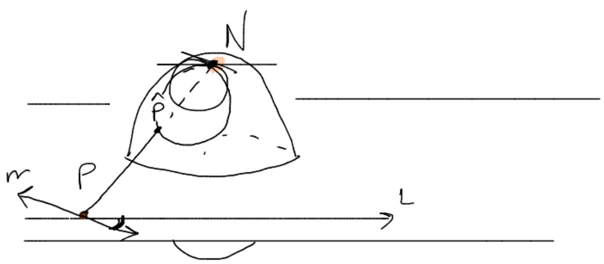




distorts angles

Stereographic Projection

preserves angles (is a conformal map)



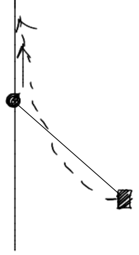
① As p moves along L the ray Np sweeps out a plane, & this plane intersects sphere in a circle

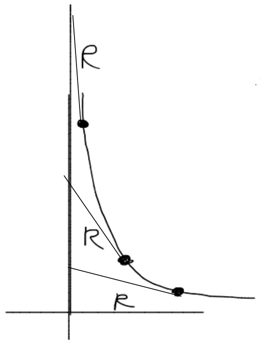
② The tangent line @ N is parallel to the line L, & the same for another line m thru P.

③ the angle in the map is the same as the angle b/w the tangents

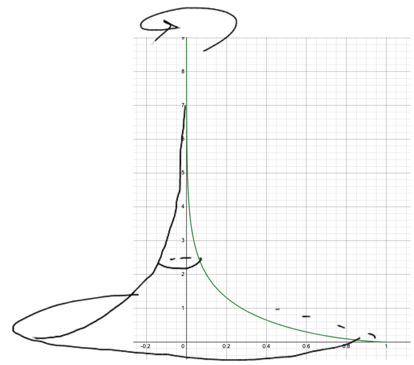
Tractrix

table



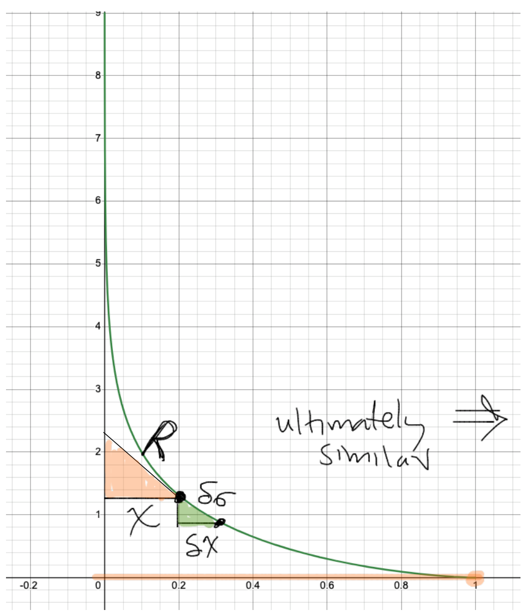


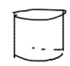
y-axis



Produce a metric on the pseudosphere

"Hyperbolic Cylinder" $\equiv$ pseudosphere.



 cylinder

coords on pseudosphere

x = "radius", horiz. distance to axis

σ = length along "generator"
vertical slice thru y-axis.

$$\frac{\delta\sigma}{\delta x} = \frac{R}{x} \Rightarrow \boxed{\frac{dx}{d\sigma} = \frac{-x}{R}}$$ separable O.D.E.

increase σ
$\Rightarrow$ decrease x

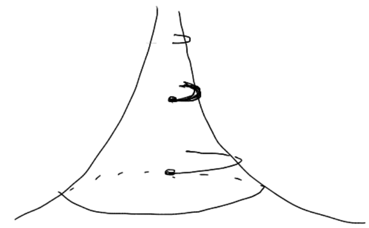
$$\frac{dx}{x} = \frac{-1}{R}d\sigma \Rightarrow \ln x = -\frac{1}{R}\sigma + C$$

$$x = e^{-\frac{\sigma}{R}+C} = e^{-\sigma/R}\cdot e^{C}$$

when $\sigma = 0$, $x = R$

$$x = Re^{-\sigma/R}$$

Metric on P. sphere



① x = angle around
σ = length up from base

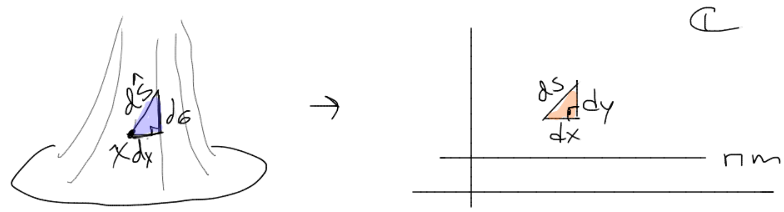
② what geometric object is:
x = constant = vert. slice
σ = constant = horiz. slice (circle)
(orthogonal coords)

③ when we increase $\underline{dx}$ $\longleftrightarrow$ $\chi dx \Rightarrow ds_1 = \chi dx$

④ Metric: $d\hat{s}^2 = ds_1^2 + d\sigma^2 = \chi^2 dx^2 + d\sigma^2 = \underbrace{\left(Re^{-\sigma/R}\right)^2}_{A} dx^2 + \underbrace{d\sigma^2}_{B=1}$

Curvature: $K = \frac{1}{-AB}\left(\partial_x \frac{\partial_x B}{A} + \partial_\sigma \frac{\partial_\sigma (A)}{B}\right) = \frac{1}{-Re^{-\sigma/R}}\left[\frac{\partial_x(-e^{-\sigma/R})}{1}\right] = \frac{1}{-Re^{-\sigma/R}}\left[-e^{-\sigma/R}\cdot(-1/R)\right] = \frac{-1}{R^2}$

Goal: Produce a conformal map on pseudosphere that is compatible w/ Hyperbolic Metric on $\mathbb{H}^2$.



(χ, σ) coords are orthogonal, our map is conformal $\Rightarrow$ Images of χ, σ will be orthog. too.

Observe! map shrinks by factor of χ. in the x direction
(mult. by $1/\chi$)

$ds_1 = \chi\, dx$
$ds_1^2 = \chi^2 dx^2$

$$\boxed{d\hat{s}^2 = \chi^2 dx^2 + \chi^2 dy^2 = \chi^2(dx^2 + dy^2) = \chi^2 ds^2}$$ (conformal) (#)

$ds_2 = \chi\, dy$
$ds_2^2 = \chi^2 dy^2$

Calculate the y-coord. for a pt. (x, σ) on pseudosphere.

$$\frac{dy}{dx} = \frac{d\sigma}{\chi\, dx} \Rightarrow \frac{dy}{d\sigma} = \frac{dx}{\chi\, dx} = \frac{1}{\chi} = \frac{1}{R e^{-\sigma/R}} = \frac{1}{R} e^{\sigma/R}$$

similarity

$$\Rightarrow y = \int dy = \int \frac{1}{R} e^{\sigma/R}\, d\sigma = \int e^u\, du = e^{\sigma/R} + C$$

$u = \sigma/R$
$du = \frac{1}{R} d\sigma$

(upper half plane: $C = 0$)

So $y = e^{\sigma/R} = \frac{R}{\chi} \Rightarrow \boxed{\chi = \frac{R}{y}}$

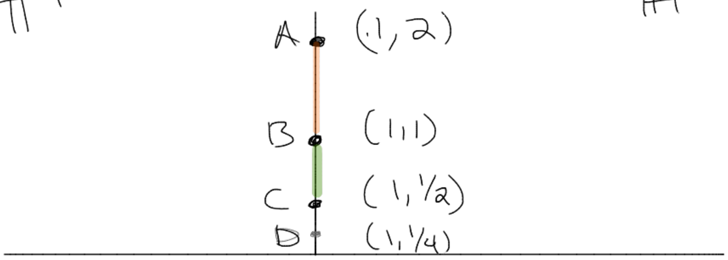
combining w/ (#)

$$\Rightarrow \boxed{d\hat{s}^2 = \left(\frac{R}{y}\right)^2 ds^2}$$

$$d\hat{s} = \frac{R}{y} ds = \frac{R\sqrt{dx^2 + dx^2}}{y}$$

In Hyperbolic geometry, $R = 1$ (typically) $\Rightarrow$ $\boxed{d\hat{s} = \frac{\sqrt{dx^2 + dy^2}}{y}}$

$\mathbb{H}^+$ $\qquad$ $\mathbb{H}^2$ $\qquad$ compute lengths along vertical lines.

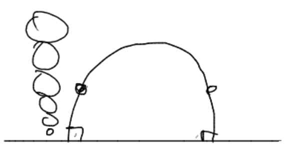


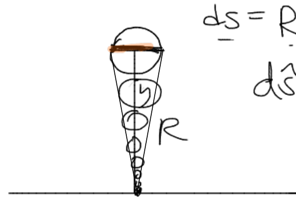
$$d(A,B) = \int_\alpha \frac{ds}{y} = \int_\alpha \frac{\sqrt{dx^2+dy^2}}{y} = \int_1^2 \frac{dy}{y} = \ln(y)\Big|_1^2 = \underline{\ln(2)}$$

$\alpha =$ vertical
$\Rightarrow dx = 0$

$$d(B,C) = \ldots = \int_{1/2}^{1} \frac{dy}{y} = \ln y \Big|_{1/2}^{1} = \ln 1 - (\ln 1 - \ln 2) = \underline{\ln(2)}$$

$$d(C,D) = \underline{\ln(2)}$$





$$ds = R\,d\theta$$

$$d\hat{s} = \frac{1}{y}ds \simeq \frac{1}{R}ds = \frac{1}{\cancel{R}}\cancel{R}\,d\theta = \underline{\underline{d\theta}}$$