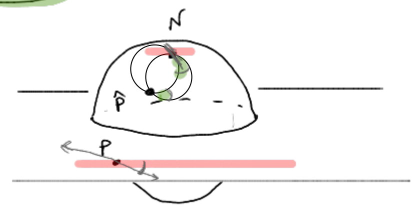
MA541 – Wk 4

① 

central Projection deforms angles

circles ⟼ ellipses

② Stereographic Projection is a conformal Map



1. As p moves along line, ray $N\hat{p}$ sweeps out a plane thru N, plane intersects sphere in a circle.

2. Tangent @ N is || to original line, & same for some other circle intersecting N. Since the angles b/w circles are exactly the angles b/w their tangents it follows.

3. So St. Proj. preserves angles.

Ch. 5

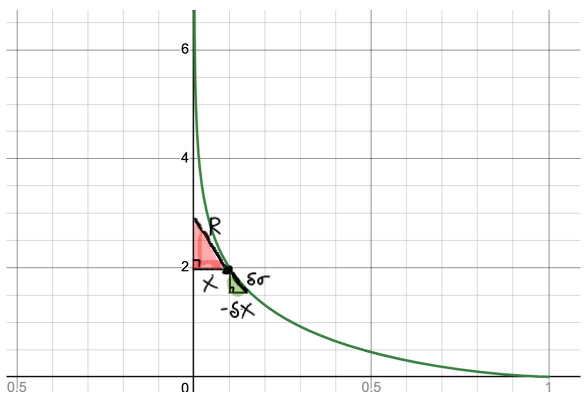
5.1 Hyperbolic Geometry is the intrinsic geometry of surface w/ constant negative curvature.

Pseudosphere:

- a surface w/ const. neg. curvature $K = -1/R^2$ → bottom rim radius
- analagous to a cylinder

5.2 Beltrami: "local geometry of psuedosphere obeys non-Eucl. geometry.

Tractrix: (from Latin "to pull or drag")



Coords: (X, σ) — σ: arc length along generator; X: horiz. dist to y-axis (radius)

$$\frac{\delta X}{\delta \sigma} = \frac{X}{R}, \Rightarrow \frac{dX}{d\sigma} = -\frac{X}{R}$$

positive change in σ ⇒ decrease in X

sep & integrate ⇒

$$\frac{dX}{X} = -\frac{1}{R}d\sigma \Rightarrow \ln(X) = -\frac{1}{R}\sigma + C$$

$$\Rightarrow X = e^{-\frac{1}{R}\sigma + C} = e^C \cdot e^{-\frac{1}{R}\sigma} \quad \& \quad X = R \text{ @ } \sigma = 0 \Rightarrow R = e^C \cdot e^0 = e^C$$

$$\boxed{X = Re^{-\sigma/R}}$$

Pseudosphere: (rotate tractrix about y-axis)

Metric & curvature:

① notation: X = angle around, σ = length up from base

② X = constant (vertical), σ = constant (horiz. circle = $Re^{-\sigma/R}$)

③ increase dX ⇔ rotation thru arc $X\,dX$ ⇒

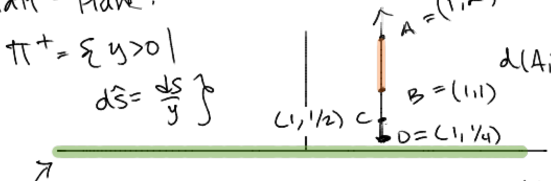
Metric $\quad d\hat{s}^2 = X^2 dX^2 + d\sigma^2 = \underbrace{(Re^{-\sigma/R})^2}_{A} dX^2 + \underbrace{d\sigma^2}_{B}$

Curvature $\quad K = \dfrac{1}{Re^{-\sigma/R}}\left[\partial_X \underbrace{\dfrac{(\partial_X 1)}{A}}_{0} + \partial_\sigma \dfrac{(\partial_\sigma (Re^{-\sigma/R}))}{B}\right] = \dfrac{1}{Re^{-\sigma/R}}\left[\dfrac{\partial_\sigma (e^{-\sigma/R})}{1}\right] = \dfrac{-\frac{1}{R} \cdot e^{-\sigma/R}}{Re^{-\sigma/R}}$

$= -1/R^2$

Half-Plane:

$\Pi^+ = \{ y > 0 \mid d\hat{s} = \frac{ds}{y} \}$



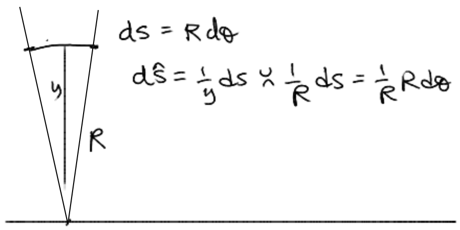
ideal boundary

vertical line: $dx = 0$

$$d(A,B) = \int_\alpha \frac{ds}{y} = \int_\alpha \frac{\sqrt{dx^2 + dy^2}}{y} = \int_1^2 \frac{dy}{y} = \ln y \Big|_1^2 = \ln 2$$

$$d(B,C) = \int_{1/2}^{1} \frac{dy}{y} = \ln y \Big|_{1/2}^{1} = \ln 2$$

$$d(C,D) = \ln 2$$



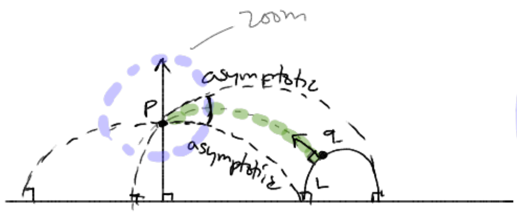
$ds = R\, d\theta$

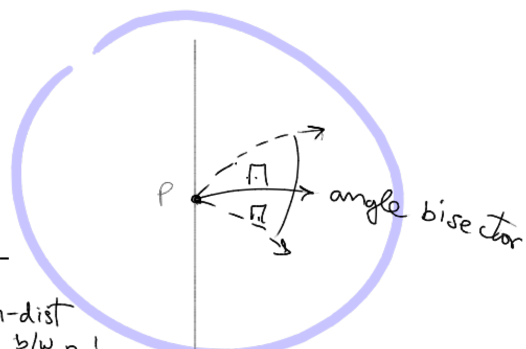
$d\hat{s} = \frac{1}{y} ds \approx \frac{1}{R} ds = \frac{1}{R} R\, d\theta = d\theta$

see P.57

Geodesics:

(generalized) great circles that intersect boundary @ $\frac{\pi}{2}$





angle ∢

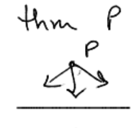
$\tan(\Pi/2) = e^{-D}$ h-dist b/w P & L

① If P close to L, $D \approx 0 \Rightarrow \tan(\Pi/2) \approx e^0 = 1 \Rightarrow \Pi/2 \approx \pi/4$

$\Rightarrow \Pi \approx \pi/2$

(Euclidean $= \frac{\pi}{2}$)

"Half the rays from P miss L"



always.

② As $D \to \infty$ $\tan(\Pi/2) \to 0$ (quickly)

$\Rightarrow \Pi/2 \to 0 \Rightarrow \Pi \to 0$

(window within which rays from P hit L)

H-Disc Model (see exercises)

$$d\hat{s} = \frac{2}{1-r^2}\, ds$$

Ch. 6 Isometries

- The set of Isom(Σ) has a 'group structure'.
- Direct Isom (preserve 'sense' $\equiv$ orientation)

HW/ lecture exercises

#12. Mercator

#23 Total curvature of torus

#27 Mobius + sphere + rotations

#26 Poincaré Disc Metric

#28 $\mathbb{H}^2 \subset \mathbb{H}^3$

#29 Curvature of $\mathbb{H}^2 \subset \mathbb{H}^3$

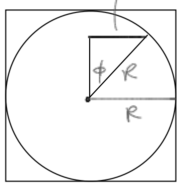
#30 Horospheres

#31 spheres in $\mathbb{H}^3$

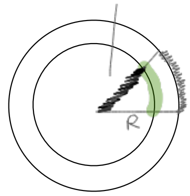
#32 Horosphere Metric

#12

(a)



V. SLICE



H. SLICE

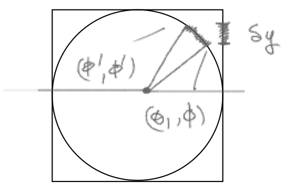
HORIZONTAL

$\boxed{d\hat{s} = R\sin\phi\, dx = R\sin\phi \frac{ds}{R} = \sin\phi\, ds}$

$ds = R\,dx \Rightarrow dx = \frac{ds}{R}$

1) & 2) ⇒ figure above

(b)



Conformal ⇒ same scale factor on horizontal curves

i.e, δs = vert. change in cylinder (i.e, δy)

if $\delta\hat{s}$ = vertical meridean arc ⇒ $\delta\hat{s} = \sin\phi\, \delta s$

or $\delta s = \frac{1}{\sin\phi} \cdot \delta\hat{s}$

$f(\phi) = y$ height of map

conformality

$f'(\phi) = \frac{df}{d\phi} \approx \frac{\delta f}{\delta\phi} \approx -\frac{\delta y}{\delta\phi} \approx -\frac{\delta s}{\delta\phi} = -\frac{1}{\sin\phi}\frac{\delta\hat{s}}{\delta\phi}$

notation

def'n of deriv.

def'n of f

+ change in φ ⇒ decrease in ht.

$= -\frac{1}{\sin\phi}\frac{R\,\delta\phi}{\delta\phi} = \frac{-R}{\sin\phi}$

(iii)

$$f(\phi) = -\int \frac{R}{\sin\phi}\,d\phi = -R\int \csc\phi\,d\phi = -R\int \csc\phi \cdot \frac{\csc\phi + \cot\phi}{\csc\phi + \cot\phi}\,d\phi$$

$$= -R\int \frac{\csc^2 + \csc\cot}{\csc \quad \cot} = R\int \frac{du}{u} = \ln|\csc\phi + \cot\phi| = R\ln\left(\frac{1}{\sin\phi} + \frac{\cos\phi}{\sin\phi}\right)$$

$$= R\ln\left(\frac{1+\cos\phi}{\sin\phi}\right)$$

$\Rightarrow$