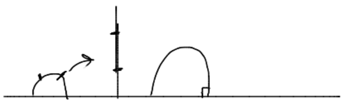
Upper half-plane model

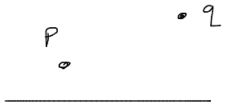
∞ Π^+



generalized circles intersecting boundary @ rt. angles

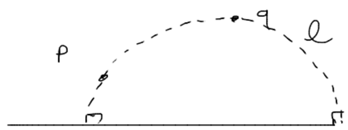
Like all "geometries", there is a group of isometries (distance preserving maps) taking (any) line to any other line.

Since it's easy to compute h-dist along vertical geodesics let's use this to compute dist b/w any pts on Π^+.



given

$\longrightarrow$

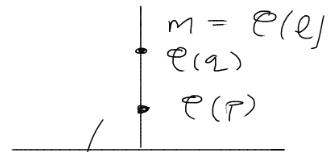


$\exists!$ geodesic thru p, q. $\rightarrow$

(linear algebra)

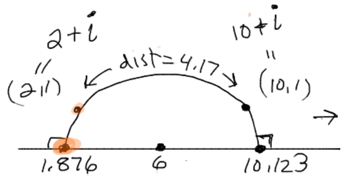
two points determine a unique circle containing them that also is centered on the x-axis

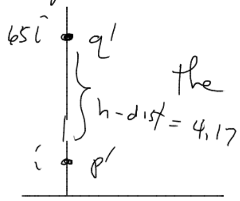
isometry φ



$$\int_{\mathrm{imag}(\varphi(p))}^{\mathrm{imag}(\varphi(q))} \frac{dy}{y} = \ln \cdot - \ln$$

Explicit computation of h-dist b/w (2,1) & (10,1)





the isometries of $\mathbb{H}^2$ are Mobius Transformations

$\varphi(z) = \dfrac{az+b}{cz+d}$ $\quad ab - dc \neq 0$

- determined exactly by their behavior on 3 points

$(x-6)^2 + (y-0)^2 = r^2$

$r = \sqrt{(10-6)^2 + (1)^2} = \sqrt{17} \approx 4.123$

$\hat{\mathbb{C}} = \mathbb{C} \cup \infty$

choose a mobius trans, s.t.
$\varphi(1.876) = 0$
$\varphi(10.123) = \infty$
$\varphi(2+i) = i$

① $\varphi(1.876) = 0 = \dfrac{a(1.876) + b}{c(1.876) + d} = 0 \Rightarrow b = -1.876a$

$\Rightarrow \varphi(z) = \dfrac{az - 1.876a}{cz - 10.123c}$

② $\varphi(10.123) = \infty = \dfrac{a(10.123) + b}{c(10.123) + d} = \dfrac{1}{0} \Rightarrow d = -10.123c$

$\varphi(z) = \dfrac{a(z - 1.876)}{c(z - 10.123)}$

Cool Fact: Mobius T's are invariant under scale!

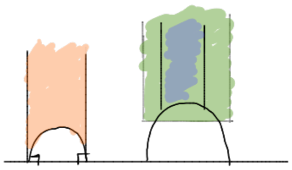
$$\frac{az+b}{cz+d} \leftrightarrow \frac{a(\frac{1}{c})z + b(\frac{1}{c})}{c(\frac{1}{c})z + d\frac{1}{c}} = \frac{\frac{1}{c}(az+b)}{\frac{1}{c}(cz+d)}$$
choose $\underline{\underline{c = 1}}$

$$\varphi(z) = a\,\frac{z - 1.876}{z - 10.123}$$

③ $\varphi(2+i) = i = a\,\dfrac{2+i - 1.876}{2+i - 10.123} = a\,\dfrac{0.124 + i}{-8.123 + i} = i$

$$a = i \cdot \frac{-8.123 + i}{0.124 + i} = \frac{-8.123i - 1}{i + 0.124} =$$

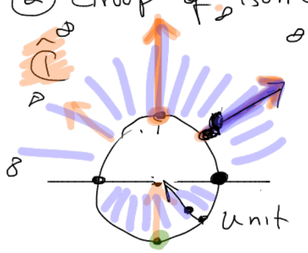
More Upper Half-Plane $\rightsquigarrow$ Disc Model



Π^+

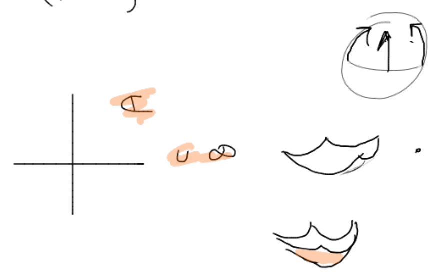
① Ideal triangles exist. (triangle whose (one or more) vertices are @ ∞)

② Group of Isometries: generated by inversions



$$\boxed{\varphi(z) = \frac{1}{z}} = \frac{0z + 1}{1z + 0} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{\det} -1 \neq 0$$

$$\varphi(i) = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i$$



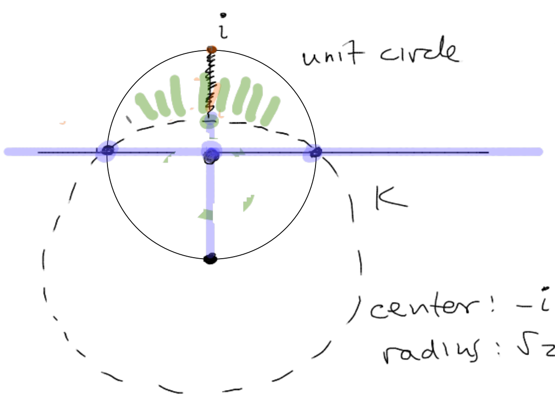
$$\varphi(1+i) = \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{1+1} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}$$

$$\varphi(\pm 1) = \pm 1$$

$$\varphi(0) = \frac{1}{0} = \infty$$

$$\varphi(\infty) = \frac{1}{\infty} = 0$$

I_K = inversion about circle K.

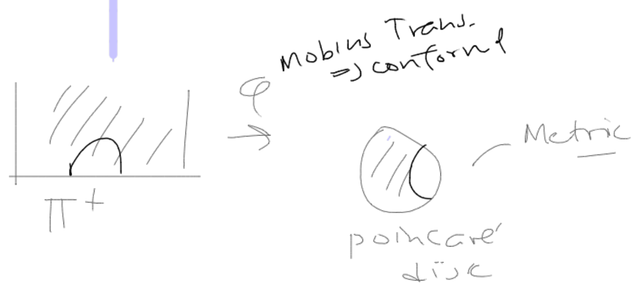


$I_K(i) = 0$

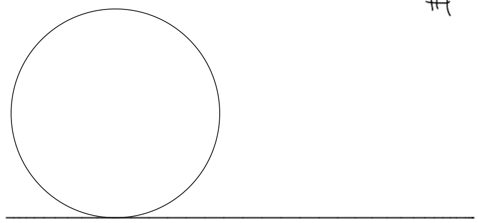
$I_K(\pm 1) = \pm 1$

Follow with $z \mapsto \bar{z}$

get $\varphi : \Pi^+ \to D =$ unit disc



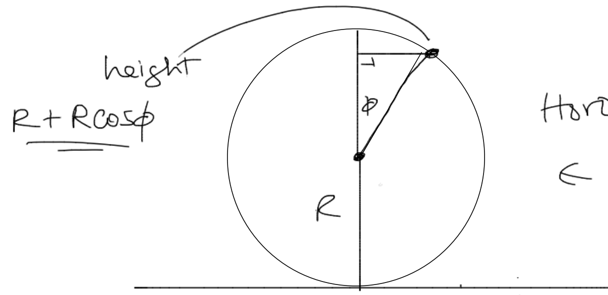
One last interesting subset of $\mathbb{H}^2$



$\mathbb{H}^2$ Horospheres! Euclidean circles in $\mathbb{H}^2$ tangent to boundary

they are flat Euclidean spaces embedded in $\mathbb{H}^2$

$\mathbb{H}^3$



Horosphere

$\leftarrow$ slice

$\mathbb{H}^3$ metric

$$d\hat{s} = \frac{ds}{\text{height}}$$

Recall metric on sphere (θ, ϕ)

$$d\hat{s}^2 = \frac{R^2 \sin^2\phi \, d\theta^2 + R^2 d\phi^2}{(R + R\cos\phi)^2} \Rightarrow d\hat{s} = \frac{R\left(\sqrt{\sin^2\phi \, d\theta^2 + d\phi^2}\right)}{R + R\cos\phi}$$

$$= A^2 d\theta^2 + B^2 d\theta^2$$

$$= \frac{\sqrt{\sin^2\phi \, d\theta^2 + d\phi^2}}{1 + R\cos\phi}$$

$A =$

$B =$