Wk 4 - Wed

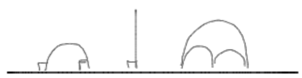
1. upper half plane model
2. isometries - mobius transformations
3. exercise: compute distance in h2
4. mobius transform that maps plane to disc
5. derive metric on disc (p. 318)
6. motions on disc
7. horosphere metric & curvature

Computational Hyperbolic Geometry Project

1. approximate the length of a path in H^2:
   a. let C be a Euclidean circle contained entirely in H^2, let U, L be its upper and lower hemispheres, respectively
   b. approximate U & V with (relatively) fine sets of discrete points
   c. approximate the pair-wise set of h-distances between adjacent points in U. repeat for V
   d. sum your U-approximations, giving an estimate of the the h-length of U. repeat for V
   e. which approximations will be longer?
   f. why does this work?

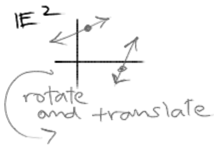
Mobius Transformations & Hyperbolic Geometry

Upper Half Plane Model of $\mathbb{H}^2$.



Like all "geometries", there is a group of isometries (distance preserving maps) (same) taking any line ℓ to any line m.

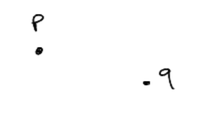
Ex $\mathbb{E}^2$



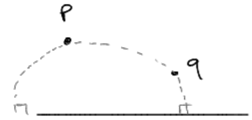
Ex

each geodesic can be mapped to any other geodesic

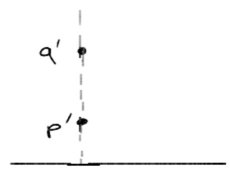
Since it's easy to compute ($\mathbb{H}$) distance b/w points lying on a vertical geodesic — we often use this algorithm to compute distance in $\mathbb{H}^2$:



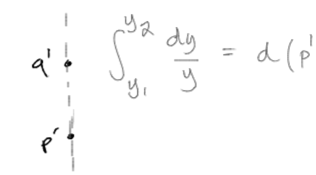
let p, q be given



$\exists!$ circle thru them whose center is on x-axis (linear algebra)



$\exists$ an isometry taking any such circle to a vertical line (Mobius Transformations)



$$\int_{y_1}^{y_2} \frac{dy}{y} = d(p', q')$$

compute distance

isometries preserve distance $\Rightarrow$

$$d(p,q) = d(p', q')$$

Ex

eg., $f(z) = \frac{az+b}{cz+d}$ w) $ad - bc = 1$

each determined uniquely by action on 3 pts $\{z_1, z_2, z_3\}$

① $z_1 = (2, 1)$ ($= 2+i$) Find z_2 another point on line:

$(x-h)^2 + (y-k)^2 = r^2$

$y = 0$ & since they are at same height

$x =$ midpoint $= \frac{10+2}{2} = 6$

$(x-6)^2 + y^2 = r^2$ and $r = \sqrt{(10-6)^2 + (1)^2}$

$\approx \sqrt{16+1} \approx 4.123$

$\Rightarrow (10.123, 0)$ is one endpoint $= z_2$

② Another: $(1.876, 0) = z_3$

want

$f(z_3) = 0$: $0 = \frac{a(1.876) + b}{c(1.876) + d} \Rightarrow -1.876a = b$

$f(z) = \frac{a(z - 1.876)}{c(z - 10.123)}$

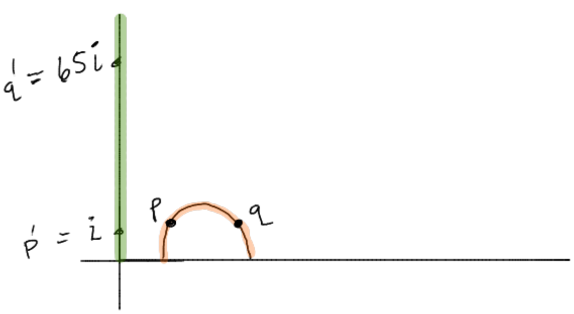
$f(z_3) = \infty$: $\frac{1}{0} = \frac{a(10.123) + b}{c(10.123) + d} \Rightarrow -10.123c = d$

$f(z) = a \cdot \frac{(z - 1.876)}{(z - 10.123)}$

(also turns out Mobius T's are invariant to scale, so multi. by C)

$f(2+i) = a \frac{(2+i) - 1.876}{(2+i) - 10.123} = a \frac{0.124 + i}{-8.123 + i} = i$

$\Rightarrow \frac{-8.123i - 1}{i + 0.124} = a$



check

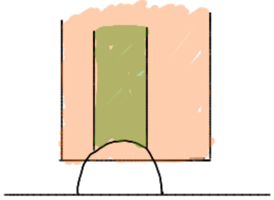
$f(2+i) \approx i$

$f(10+i) \approx -0.6 + 65i$

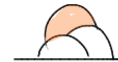
↑ ignore:

$\int_1^{65} \frac{dy}{y} = \ln 65 - \ln 1 \approx \ln 65 \approx 4.17$

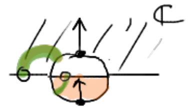
More Upper Half-Plane



$\mathbb{H}^2$

1. Ideal triangles exist.
   (as do non-ideal ones 

2. Group of isometries are generated inversions



$f(z) = \frac{1}{z} = \frac{0z+1}{1z+0} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Inversion
$I_u =$ in unit circle

$f(i) = \frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = -i$

$f(\infty) = \frac{1}{\infty} = 0$

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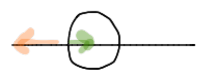
unit circle

I_K - inversion sends $i \mapsto 0$
$+1 \mapsto +1$
$-1 \mapsto -1$

Followed by $z \mapsto \bar{z} \Rightarrow$
$(0 \mapsto 0)$
$+1 \mapsto -1$
$-1 \to -1$

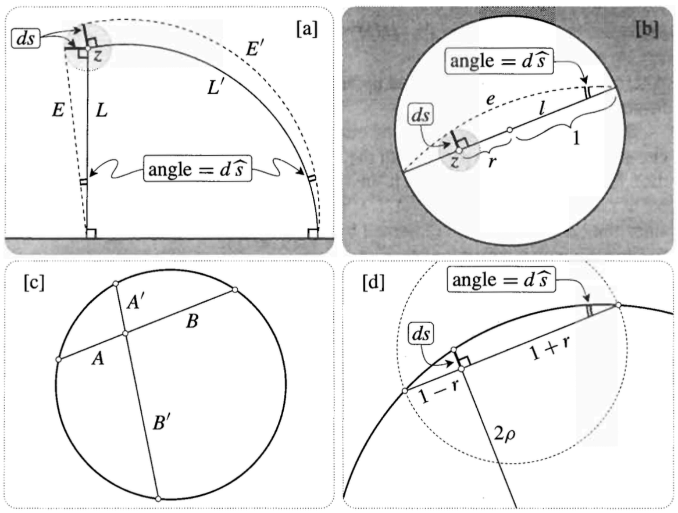


$$\varphi(z) = \frac{iz - 1}{z + 1}$$

sending $\Pi^+ \to$ Disc
(thus embuing hyperbolic metric on Disc)

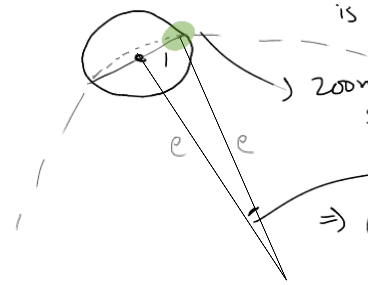
Next, let's find metric on D.

1. If ds is infinitessimal Euclidean length of horizontal line element emanating from z, then angle b/w L & E is its hyperbolic length $d\hat{s} = \frac{ds}{\mathrm{Im}(z)}$



Since mobius Trans is conformal this also holds in D

choose ds ⊥ line l, since ds is infinitesimal circle thru arc e is huge



zoom to see similar △'s so $d\hat{s}$

$\Rightarrow \rho d\hat{s} = 1$

$2\rho ds = (1-r)(1+r) = 1 - |z|^2 \Rightarrow$ $d\hat{s} = \frac{2}{1-|z|^2} ds$