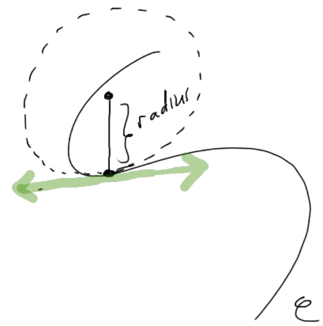
Curvature of Plane Curves

same intrinsic geometry

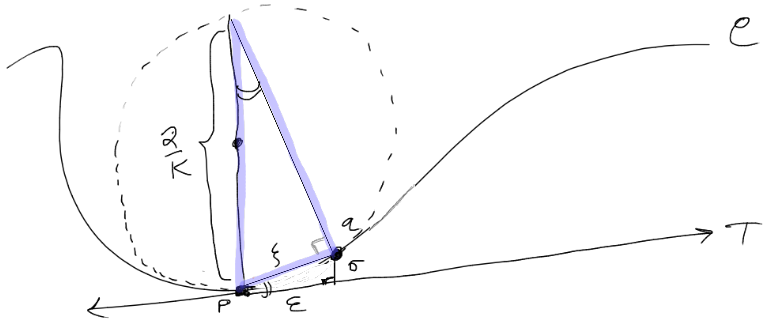


curvature of 1-D is extrinsic

Newton: "crookednesse", "curvature" $= \frac{1}{\text{radius}} = K$

circle of curvature

K: how fast the curve departs from its tangent



$\lim_{\varepsilon \to 0} \frac{\sigma}{\varepsilon} = 0$

thus

$$\frac{\xi^2}{\varepsilon^2} = \frac{\sigma^2 + \varepsilon^2}{\varepsilon^2} = \left(\frac{\sigma}{\varepsilon}\right)^2 + 1 \asymp 0 + 1 = 1$$

$$\Rightarrow \xi \asymp \varepsilon$$

Similar Δ's

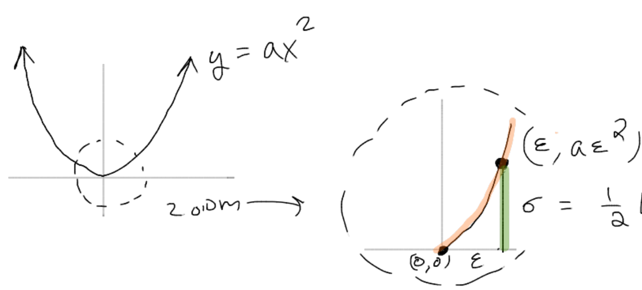
$$\frac{\text{base}}{\text{hyp}} = \frac{\xi}{(2/K)} \asymp \frac{\sigma}{\xi}$$

$$\frac{K\xi}{2} \asymp \frac{\sigma}{\xi}$$

$$K \asymp \frac{2\sigma}{\xi^2} \asymp \frac{2\sigma}{\varepsilon^2}$$

$$\sigma = \frac{1}{2} K \varepsilon^2$$

Ex



$\sigma = \frac{1}{2} K \cancel{\varepsilon^2} = a \cancel{\varepsilon^2}$

$K = 2a$

$K = \frac{1}{r}$

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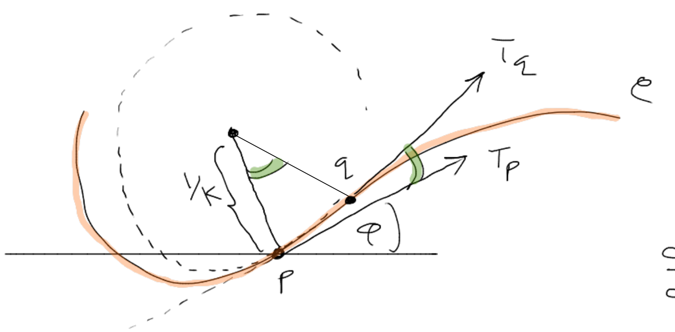
100 years later:

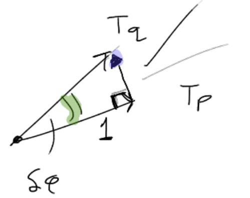
Kaestner: Curvature is the rate of turning of the tangent w.r.t. arc length

If φ is the angle the tangent makes w/ horizontal & s = arc length of curve then $K = \frac{d\varphi}{ds}$

Next, relate κ to tangent & normal vectors



a normal vector



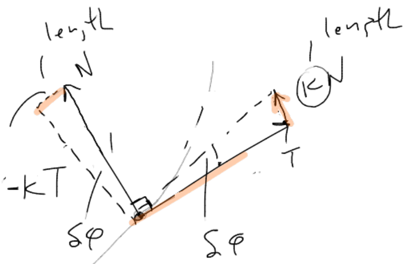
$\delta T \approx \delta\varphi \cdot N$

$$\frac{dT}{ds} \approx \frac{\delta T}{\delta s} \approx \frac{\delta\varphi \cdot N}{\delta s} \approx \frac{\delta\varphi}{\delta s} \cdot N \approx \kappa N$$

$$\boxed{\frac{dT}{ds} = \kappa N}$$

ASIDE

$\frac{\delta\varphi}{\delta s} \approx \kappa$ make sense since for a full rotation about the circle of curvature $\delta\varphi \approx 2\pi$, $\delta s \approx 2\pi \cdot (1/\kappa)$:)



$\delta N = -\kappa T$

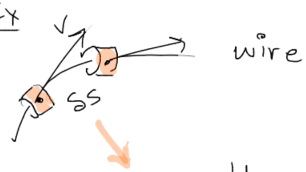
$$\boxed{\frac{dN}{ds} = -\kappa T}$$

$\delta N = \delta\varphi \cdot (-T)$

since $|N| = 1$

$\delta\varphi = \text{arclength } \kappa$

Ex



s = arc length

bead moving on wire, mass m, w/ constant speed v

$\delta s = v\,\delta t$

$\bar{v} = v\bar{T}$

$$F = ma = m\frac{d\bar{v}}{dt}$$

$$= m\frac{d}{dt}(v\bar{T}) = mv\frac{d\bar{T}}{dt}$$

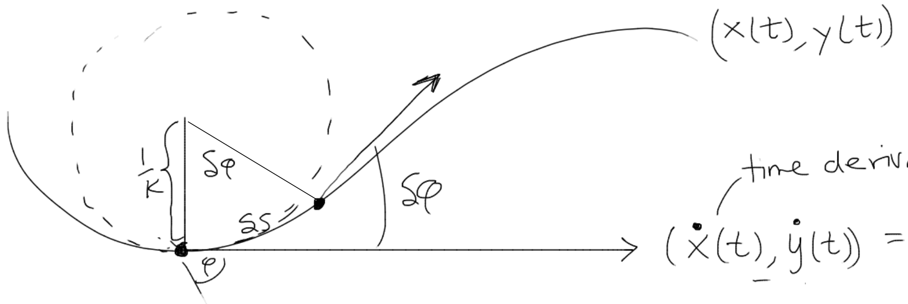
chain rule

$$= mv\frac{d\bar{T}}{ds} \cdot \underbrace{\frac{ds}{dt}}_{=v}$$

$$= mv^2\frac{d\bar{T}}{ds} = mv^2\kappa \cdot N$$

For parametric curves

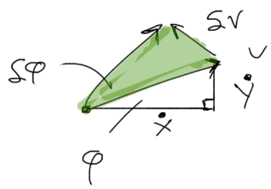
Next, derive a formula for κ in terms of derivatives.



time derivative, $(x' \sim$ wrt. arc length$)$

$(\dot{x}(t), \dot{y}(t)) = \bar{v}$

Bring two tang. vectors together



$v = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}$, $\tan(\varphi) = \frac{\dot{y}}{\dot{x}}$

Plan: Take time deriv. of ↑, isolate $\dot{\varphi} = \frac{d\varphi}{dt}$ relate that to $\frac{d\varphi}{ds} = \kappa$

$$\frac{d}{dt}\left(\tan(\varphi)\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)$$

$$\parallel$$

use △

$$\sec^2(\varphi)\cdot\frac{d\varphi}{dt} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^2} \Rightarrow \frac{d\varphi}{dt} = \cos^2(\varphi)\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2} = \frac{\cancel{\dot{x}^2}}{\dot{x}^2 + \dot{y}^2}\cdot\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\cancel{\dot{x}^2}}$$

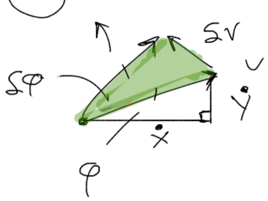
chain rule

$$\frac{d\varphi}{ds}\cdot\underbrace{\frac{ds}{dt}}_{\substack{v = |\bar{v}| \\ \text{(speed)} \\ = \sqrt{\dot{x}^2 + \dot{y}^2}}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}$$

$$\kappa = \frac{d\varphi}{ds} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\bar{v}^3}$$

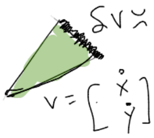
Hidden "area" in κ formula.

(∠) area of sector $= \frac{1}{2}r^2\cdot\theta$



Area: $= \frac{1}{2}\underbrace{(\dot{x}^2 + \dot{y}^2)^2}_{v^2}\;\underbrace{\delta\varphi}_{\kappa\delta s\, =\, v\delta t}$

$= \frac{1}{2}v^3\kappa\,\delta t$ (A)



$\delta v \simeq \dot{v}\delta t = \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}\delta t$

$$\rightarrow \text{Area} = \frac{1}{2}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \times \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}\delta t = \frac{1}{2}\overset{\kappa v^3}{\overbrace{(\dot{x}\ddot{y} - \dot{y}\ddot{x})}}\,\delta t = \frac{1}{2}\kappa v^3\,\delta t$$