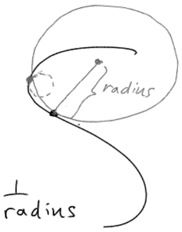
Curvature of Plane Curves

same intrinsic geometry

curvature of 1-D objects is extrinsic

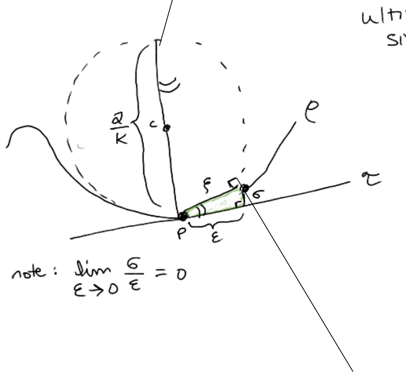
Newton: "crookednesse" : circle of curvature
tangent line
radius of curvature
κ = "crookednesse" ... "curvature" $= \frac{1}{\text{radius}}$



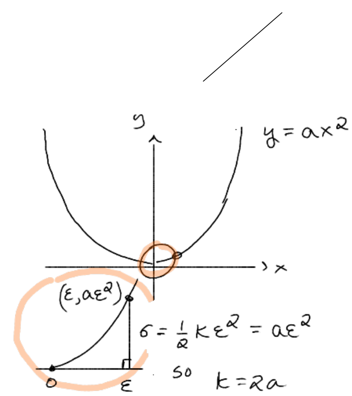
κ: How fast does the depart from its tangent line?

thus $\frac{\xi^2}{\varepsilon^2} = \frac{\varepsilon^2 + \sigma^2}{\varepsilon^2} = 1 + \left(\frac{\sigma}{\varepsilon}\right)^2 \simeq 1 \Rightarrow \xi \simeq \varepsilon$

ultimately similar Δ's : $\frac{\xi}{(2/\kappa)} \simeq \frac{\sigma}{\xi} \Rightarrow \frac{\kappa \xi}{2} \simeq \frac{\sigma}{\xi}$, $\kappa \simeq \frac{2\sigma}{\varepsilon^2}$, $\sigma = \frac{1}{2}\kappa\varepsilon^2$



note: $\lim_{\varepsilon \to 0} \frac{\sigma}{\varepsilon} = 0$



$\sigma = \frac{1}{2}\kappa\varepsilon^2 = a\varepsilon^2$

so $\kappa = 2a$

ex circle of curvature @ origin

$a = 1$

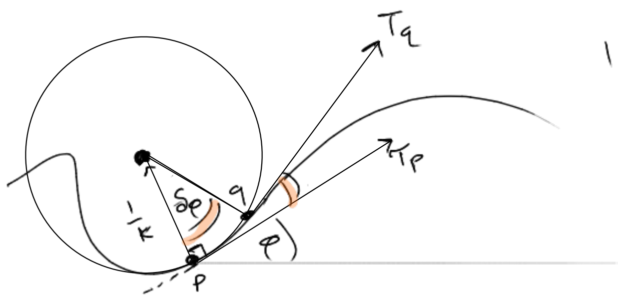
Curvature as Rate of Turning:

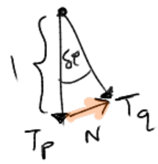
Kaestner: Curvature is the rate of turning of the tangent with respect to arc length. i.e.,

If φ is the angle of the tangent, s the arc length

then $\boxed{\kappa = \frac{d\varphi}{ds}}$

Next, relate κ to tangent & normal vectors



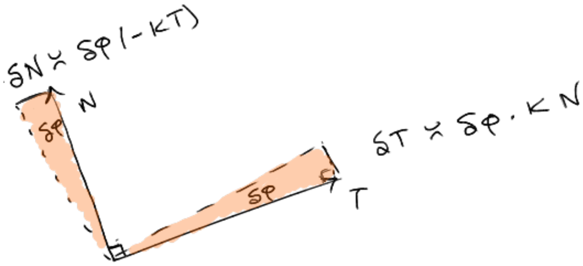


$$\delta T \asymp \delta\varphi \cdot N$$

$$\frac{dT}{ds} \asymp \frac{\delta T}{\delta s} \asymp \frac{\delta\varphi \cdot N}{\delta s} \asymp \kappa N$$

Note:

$\frac{\delta\varphi}{\delta s} \asymp \kappa$ makes sense since for a full rotation around circle of curvature : $\delta\varphi = 2\pi$, $\delta s = 2\pi \cdot \left(\frac{1}{\kappa}\right)$ so $\frac{\delta\varphi}{\delta s} = \kappa$:)



$$\boxed{\frac{dT}{d\varphi} = \kappa N \qquad \frac{dN}{d\varphi} = -\kappa T}$$

Ex bead, mass m, moving w/ constant speed v on friction-less wire:

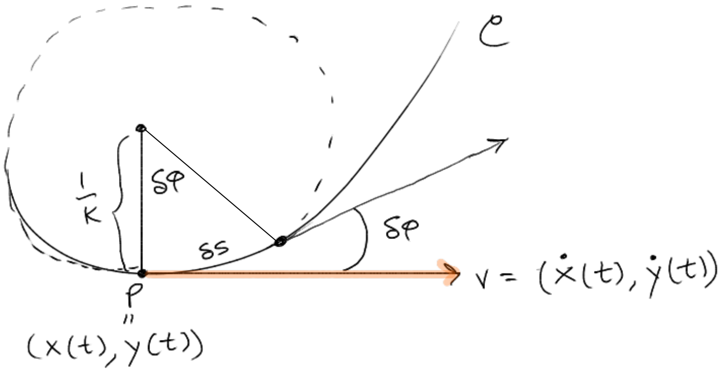
$\delta s = v\,\delta t$

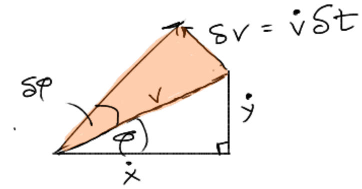
$\bar{v} = v\bar{T}$

$$\bar{F} = m\bar{a} = m\frac{d}{dt}(\bar{v}) = m\frac{d}{dt}(v\bar{T}) = mv\frac{d\bar{T}}{dt}$$

$$= mv \cdot \frac{dT}{ds} \cdot \frac{ds}{dt} = mv\,\kappa N \cdot v = \kappa m v^2 N$$

Next Curvature Formula





$$\bar{v} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \quad \text{and} \quad \tan\varphi = \frac{\dot{y}}{\dot{x}}$$

$$\frac{d}{dt}(\tan\varphi) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}$$

$$= \sec^2\varphi \cdot \dot{\varphi}$$

$$\dot{\varphi} = \cos^2\varphi \, \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2} = \frac{\cancel{\dot{x}^2}}{\dot{x}^2 + \dot{y}^2} \cdot \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\cancel{\dot{x}^2}}$$

$$= \frac{d\varphi}{dt} = \frac{d\varphi}{ds} \cdot \frac{ds}{dt}$$

$$\kappa \cdot \frac{ds}{dt} = \kappa \cdot v = \kappa\sqrt{\dot{x}^2 + \dot{y}^2}$$

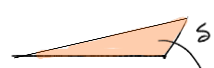
$$\text{so} \quad \kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

$$\boxed{\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{v^3}}$$

$$\left.\begin{aligned} \delta\varphi &\approx \kappa\,\delta s \\ \delta s &\approx v\,\delta t \end{aligned}\right\} \quad \begin{aligned} \delta v &\approx \delta\varphi \cdot |v| \\ &\approx \delta\varphi\sqrt{\dot{x}^2 + \dot{y}^2} \end{aligned}$$

$$\text{Area sector} = \frac{1}{2} r^2 \theta$$

$$\begin{aligned} \text{Area} &\approx \frac{1}{2}(\dot{x}^2 + \dot{y}^2)\,\delta\varphi \\ &\approx \frac{1}{2}(\dot{x}^2 + \dot{y}^2)\,\kappa\,\delta s \approx \frac{1}{2}v^2\,\delta\varphi \\ &\approx \frac{1}{2}(\dot{x}^2 + \dot{y}^2)\,\kappa v\,\delta t \\ &\approx \frac{1}{2}(\dot{x}^2 + \dot{y}^2)^{3/2}\,\kappa\,\delta t \\ &\approx \frac{1}{2}v^3 \kappa\,\delta t \quad ★ \end{aligned}$$



$$\delta v \approx \dot{v}\,\delta t \approx \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} \delta t$$

$$v = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

Area = $\frac{1}{2}$ cross product

$$A = \frac{1}{2}(\dot{x}\ddot{y} - \dot{y}\ddot{x})\,\delta t$$

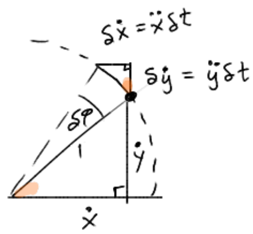
$$= \frac{1}{2}(v^3 \kappa)\,\delta t$$

recover ★

If particle moves @ unit speed:

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = 1 \iff \dot{x}^2 + \dot{y}^2 = 1$$

then $\delta s = \delta t$

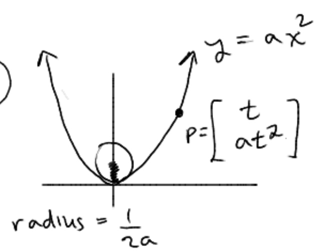


$$\frac{\delta\varphi}{\ddot{y}\,\delta t} \approx \frac{1}{\dot{x}} \Rightarrow \frac{d\varphi}{dt} = \kappa = \frac{\ddot{y}}{\dot{x}}$$

and since $\dot{x}, \dot{y}$ are $\perp$ unit circle

$$\kappa = \frac{-\ddot{x}}{\dot{y}}$$

ex



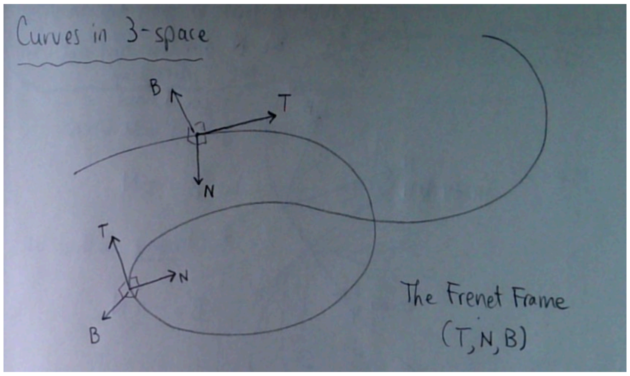
$$\text{velocity} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 2at \end{bmatrix}, \quad \text{accelleration} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 2a \end{bmatrix}$$

$$\text{speed } v = \sqrt{1^2 + (2at)^2} = \sqrt{1 + 4a^2t^2}$$

$$\kappa = \frac{1\cdot 2a - 2at\cdot 0}{{}^{3}\sqrt{1+4a^2t^2}} = \frac{2a}{{}^{3}\sqrt{1+4a^2t^2}}, \quad \frac{@\ t=0}{\kappa = 2a} \quad \Bigg| \quad \frac{@\ t=1}{\kappa = \frac{2a}{1+4a^2}}$$

observe! as $t \to \infty$ $\quad \kappa = \frac{2a}{(1+\infty)^{3/2}} = 0$

Curves in 3-space



$$T' = kN$$

$$N = \frac{T'}{|T'|}$$

curvature vector

$$\bar{K} = kN$$

$$B = \frac{T \times T'}{|T'|} = T \times \frac{T'}{|T'|} = T \times N$$

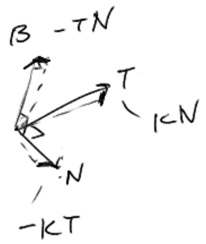
$$B' = -\tau N$$

τ = torsion, rate of turning of B about T

$$T = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad N = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\begin{matrix} i & j & k \\ v_1 & v_2 & v_3 \\ n_1 & n_2 & n_3 \end{matrix} \qquad T \times N = \begin{bmatrix} v_2 n_3 - n_2 v_3 \\ v_1 n_3 - n_1 v_3 \\ v_1 n_2 - n_1 v_2 \end{bmatrix} = B$$

$$\dot{B} = \begin{bmatrix} \dot{v}_2 n_3 + v_2 \dot{n}_3 - \dot{n}_2 v_3 - n_2 \dot{v}_3 \\ ---- \\ ----- \end{bmatrix}$$



Note: B tips only in N-direction, not T

$\dot{T} = \kappa N$

$\dot{N} = -\kappa T + \tau B$ ($\dot{N}$ needs to be @ right angles w/ N

$\dot{B} = -\tau N$

i.e., $\dot{N} \in \text{span}(T, B)$

& in fact $\dot{N} = -\kappa T + \tau B$

Frenet - Serret Equations

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \Omega \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Ω = skew symmetric 3×3 $(\Omega^T = -\Omega)$