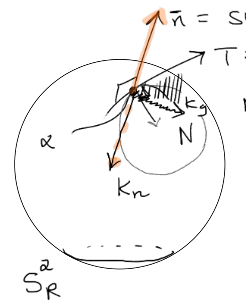
In Desmos, produce a dynamic illustration of the circle of curvature of an ellipse.

1. Parameterize an ellipse (it can be centered at the origin)
2. Use the formula for curvature of parametric curves we derived in class to get a formula for the radius of curvature of parametric curves.
3. Express the center of the circle of curvature in the the coordinates of your parameterization.

wk 6 - Mon

Today: Ch. 10, 11, 13

Geodesic Curvature Example



$\bar{n}$ = surface normal

$T = \alpha'(0)$ velocity

$N = \frac{1}{k}\alpha''(0)$ acceleration (N may not be in direction of $\bar{n}$)

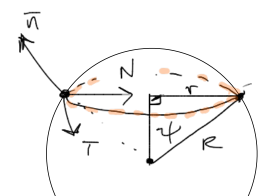
curvature

$\bar{k}_n$ = normal curvature vector $= k_n \bar{n}$

(perpendicular to this is

$\bar{k}_g$ = geodesic curvature vector , span the tangent plane T_p.

ex



$\Pi = \{ (x,y,c) , \cdot c = 1/2, \ x,y \in \mathbb{R} \}$, $\alpha = \Pi \cap S^2_R$ curve

$r = R\sin\psi$

$\alpha = (r\cos\theta, r\sin\theta, \frac{1}{2})$

$\alpha' = (-r\sin\theta, r\cos\theta, 0)$

$T = \frac{1}{r}\cdot\alpha'$

$\alpha'' = (-r\cos\theta, -r\sin\theta, 0)$

$N = \frac{1}{r}\alpha''$

$\bar{n}$ = on ray based @ $(0,0,0)$

surface normal

$\bar{n} = (r\cos\theta, r\sin\theta, r) = r(\cos\theta, \sin\theta, 1)$

$\bar{n}\cdot\alpha' = -r^2\cos\theta\sin\theta + r^2\cos\theta\sin\theta + r\cdot 0 = 0$

and $\alpha''\cdot\bar{n} = -r^2\cos^2\theta - r^2\sin^2\theta + 0 = -r^2 = -(R\sin\psi) = -R^2\sin^2\psi$

as $\psi \to 0$ $\alpha''\cdot\bar{n} \to 0$

near north pole, the curvature vector is nearly $\perp$ surface normal

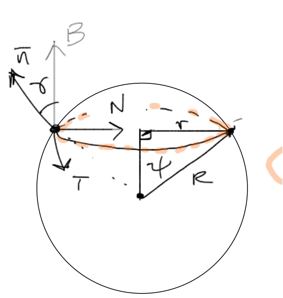
so since $\underset{\text{curvature of } \alpha}{\bar{K}} = \bar{K}_n + \bar{K}_g$, $\bar{K}_n \approx 0$ so nearly all of the curvature of α is geodesic.

on the other hand

if $\psi \to \pi/2$, $\alpha \to$ equator

as α becomes more geodesic, it's geodesic curvature vanishes

$\alpha''\cdot\bar{n} \to -R^2\cdot\underbrace{\sin^2(\pi/2)}_{\to 1} = -R^2 \Rightarrow \bar{K} \approx \bar{K}_n$ i.e., $\bar{K}_g \to 0$



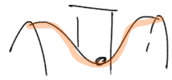
(i.e., $K_g = K\cdot\cos\gamma$

$K_n = K\sin\gamma$

$\gamma = \psi$

so $K_g \to K\cos\frac{\pi}{2} \to 0$

Review of different notions of curvature



Newton
1664

$K = \frac{1}{r}$

radius of curvature

(curves)

Kaestner

$K = \frac{d\varphi}{ds}$

rate of turning

(curves)

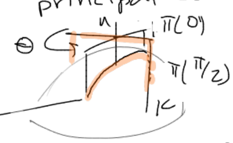
Euler
1760

set $\theta = 0 \leftrightarrow$ principal direction

$K(\theta) = K_1 \cos\theta + K_2 \sin\theta$

(surfaces)

K_1, K_2 (max, min)
principal curvatures



normal section

θ = angle b/w plane & arb. direction.

Gauss
1827

(intrinsic)

$K \approx \frac{\varepsilon(\Delta)}{A(\Delta)}$ angle defect per unit area

(surfaces)

(extrinsic)

$K = \frac{\delta \tilde{A}}{\delta A}$ mag. factor of spherical map

(formula)

$K_{ext} = K_1 \cdot K_2$

Π_θ = plane thru $\bar{n}$ making angle w/ arb direction.

Intersect Π_θ w/ S giving normal sections, compute curvature of each section.

1-D

κ_1, κ_2 max, min,

Choose θ = one principle direction.

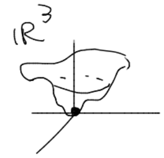
then $\kappa(\theta) = \kappa_1 \cos\theta + \kappa_2 \sin\theta = \bar{\kappa} + \frac{\Delta\kappa}{2}\cos(2\theta)$, $\bar{\kappa} = \frac{\kappa_1 + \kappa_2}{2}$

mean curvature

$\updownarrow$

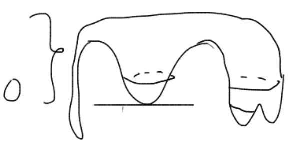
$\bar{\kappa} = 0 \leftrightarrow$ minimal surfaces

1. Assume: surface tangent to $(0,0,0)$
   x, y axes $\in T_p$



$\mathbb{R}^3$

2. Surface $\{(x,y,z) \mid z = f(x,y)\}$
   $f(0,0) = 0$, $\partial_x f(0,0) = f_x(\bar{0}) = 0$, and $f_y(\bar{0}) = 0$



3. Recall: In 1-D

Taylor Series of f @ $x=a$: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots$

2-D @ (u,v)

$$f(x,y) = f(u,v) + f_x(u,v)(x-u) + \underbrace{\frac{f_{xx}(u,v)}{2!}}_{=a}(x-u)^2 + \ldots$$
$$+ f_y(u,v)(y-v)$$
$$+ \underbrace{2\frac{f_{xy}(u,v)}{2!}}_{c=}(x-u)(y-v)$$
$$+ \underbrace{\frac{f_{yy}(u,v)}{2!}}_{=b}(y-v)^2$$

For us, $(u,v) = 0$, 1st partials die

4. Ultimately $f(x,y) = z = ax^2 + by^2 + cxy$

5. $\{(x,y,z)\} \Rightarrow$ slices are conic sections, set z = const.

6. Every conic has 2 axes of symmetry.

7. $\therefore$ the surface, locally has two axes of mirror symmetry.
   these are the principal directions

8. change coords $x, y \Leftrightarrow$ symmetry directions

(9) under this coord change $x \mapsto -x$, $y \mapsto -y$ preserve equation

$\Rightarrow c = 0$

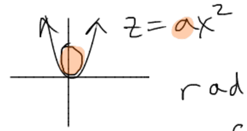
$$f(x,y) = ax^2 + by^2 = z$$

(10) w/ $\theta = 0 \leftrightarrow$ x-axis

$k_1 = k(0) =$ curvature of $\Pi_0 \cap S = C_0 \leftarrow$ what kind of curve

Π_0 | xz plane

So $y = 0$ $\quad ax^2 = z$



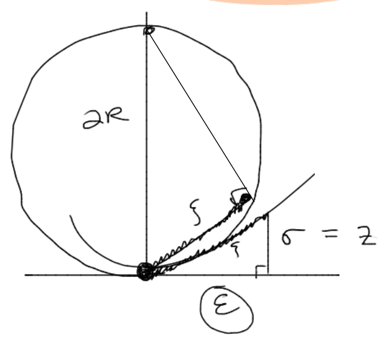
radius of curvature: $\quad a = \frac{1}{2}k_1$

Same for $\theta = \frac{\pi}{2} \leftrightarrow$ y-axis ... $by^2 = z$ ... $\rightarrow$ $b = \frac{1}{2}k_2$

(11) $z = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2$

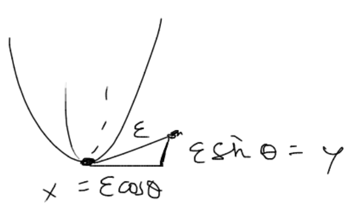
(height)

we saw



$$k(\theta) = 2\left(\frac{\overset{=z}{\sigma}}{\varepsilon^2}\right) = 2\left(\frac{\frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2}{\varepsilon^2}\right) = \frac{k_1 x^2 + k_2 y^2}{\varepsilon^2}$$

(12)



$$k(\theta) = \frac{k_1(\varepsilon\cos\theta)^2 + k_2(\varepsilon\sin\theta)^2}{\varepsilon^2}$$

$$= k_1\cos^2\theta + k_2\sin^2\theta$$

points: the principal directions are pi/2 apart….plot the function