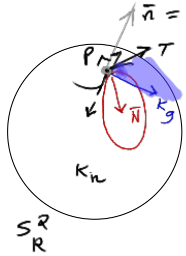
Wk 6 Mon

Today: Ch. 10, 11, 13

Geodesic Curvature Example



$\bar{n}$ = surface normal

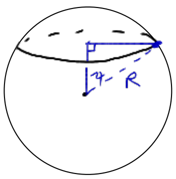
T = tangent to curve = $\alpha'(0)$ (velocity)

$\bar{N}$ = accelleration = $\alpha''(0)$

$\bar{\kappa}_n$ = normal curvature

$\bar{\kappa}_g$ = geodesic curvature

Ex $S = S^2_R$, $\Pi = \{x, y, c\}$ w/ $c = 1/2$, $x, y \in \mathbb{R}$, $\alpha = \Pi \cap S$.

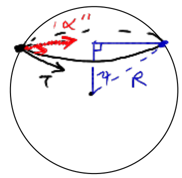


$r = R\sin\psi$

$\alpha = (r\cos\theta, r\sin\theta, 1/2)$

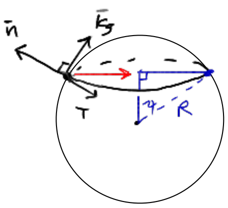
$\alpha' = (-r\sin\theta, r\cos\theta, 0)$

$T = \frac{1}{r}\cdot\alpha'$



$\alpha'' = (-r\cos\theta, -r\sin\theta, 0)$

$N = \frac{1}{r}\alpha''$



$\vec{n}$ lies on a ray emanating from $(0,0,0)$ so.

$n = (r\cos\theta, r\sin\theta, r) = r(\cos\theta, \sin\theta, 1)$ & $n\cdot\alpha' = 0$.

and

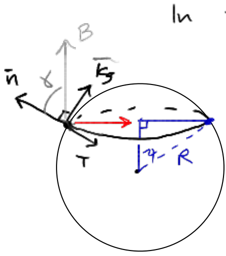
$\alpha''\cdot\vec{n} = -r^2\cos^2\theta - r^2\sin^2\theta + 0 = -r^2(\cos^2\theta + \sin^2\theta) = -r^2$

recall

$r = R\sin\psi$, so $\alpha''\cdot\vec{n} = R^2\sin^2\psi$

as $\psi \to 0$, $\alpha''\cdot\vec{n} \to 0$ meaning very little of the curvature is normal curvature.

In this case the binormal B to α ($B \perp T$ and $B \perp N$) is vertical.



So $\gamma = \psi$ and evidently,

$\kappa_g = \kappa\cos\gamma$, $\kappa_n = \kappa\sin\gamma$ w/ γ as shown

$\Rightarrow$ as $\psi \to \pi/2$, $\underbrace{\kappa_g \to 0}_{\text{approaching a geodesic}}$, $\kappa_n \to \kappa$

Goal: Prove Euler's curvature Formula

First, let us sort out what "curvature" means here:

Newton

1664

$\kappa = \frac{1}{r}$

r = radius of curvature

"Newtonian Curvature"

(curves)

Kaestner **Euler**

1761 1760

$\kappa = \frac{d\varphi}{ds}$

(curves)

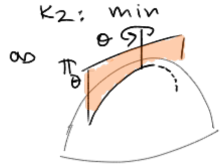
$\kappa(\theta) = \kappa_1 \cos\theta + \kappa_2 \sin\theta$

(surfaces)

principal curvature

κ_1 : max
κ_2 : min
values of curvature of "sections"

curves = $S \cap \Pi$



Gauss

1827

(intrinsic)

Gaussian Curvature

$K \simeq \frac{\varepsilon(\Delta)}{A(\Delta)}$ angle defect per unit area

(surfaces)

(extrinsic)

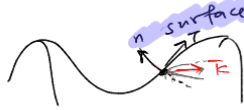
$K_{ext} \simeq \frac{\delta \tilde{A}}{\delta A}$ local area mag. factor of Spherical Map

Formula: $K_{ext} = \kappa_1 \cdot \kappa_2$

curvature vector

$\bar{\kappa} = \kappa N$

surface normal



$\bar{\kappa}$ decomposes into

$\bar{\kappa} = \bar{\kappa}_g + \bar{\kappa}_n$

geodesic curvature vector — normal curvature vector

tangent plane — direction of surface normal

Π_θ = plane (thru $\bar{n}$) making angle θ w/ arbitrary direction, intersects S in normal sections, we compute curvature of each.

κ_1, κ_2 are max, min.

Choose direction to be one of the 'principal directions', i.e., $\theta = 0$.

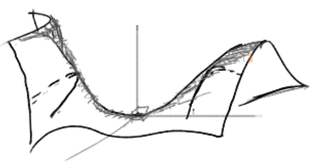
then $\kappa(\theta) = \kappa_1 \cos\theta + \kappa_2 \sin\theta$

$= \bar{\kappa} + \frac{\Delta\kappa}{2}\cos(2\theta)$ w) $\bar{\kappa} = \frac{\kappa_1+\kappa_2}{2}$ mean curvature

example of a surface w/ mean curvature $\equiv 0$

link

proof ① Assume surface is in $\mathbb{R}^3$, tangent to $(0,0,0)$ w/ x,y axes $\in T_p$



② Surface $= \{(x,y,z) \mid z = f(x,z)\}$

$f(0,0) = 0$, $\partial_x f(\bar{0}) = f_x(\bar{0}) = 0$, & $f_y(\bar{0}) = 0$.

③ Taylor Expansion of $f(x,y)$ @ $(\underset{u}{\alpha}, \underset{v}{\beta})$: First recall Taylor in 1-D

$f(x) = f(\underset{u}{\alpha}) + f'(\underset{u}{\alpha})(x-\underset{u}{\alpha}) + f''(\underset{u}{\alpha})\frac{(x-\underset{u}{\alpha})^2}{2!} + \ldots$

then $f(x,y) = f(\overset{u,v}{\alpha,\beta}) + f_x(\overset{u,v}{\alpha,\beta})(x-\overset{u}{\alpha}) + \overset{A}{f_{xx}}(\overset{u,v}{\alpha,\beta})\frac{(x-\overset{u}{\alpha})^2}{2!} + \ldots$

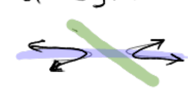
$+ f_y(\underset{u,v}{\alpha,\beta})(y-\underset{v}{\beta})$ $+ 2f_{xy}(\overset{u,v}{\alpha,\beta})\frac{(x-\overset{u}{\alpha})(y-\overset{v}{\beta})}{2!}$ (B)

$+ f_{yy}(\overset{u,v}{\alpha,\beta})\frac{(y-\underset{v}{\beta})^2}{2!}$ (C)

④ $(\overset{u,v}{\alpha,\beta}) = (0,0)$, $f_x = f_y = 0$ $\Rightarrow$ $f(x,y) = z = ax^2 + by^2 + cxy$

⑤ slices are conics:

⑥ Conics have two axes of symmetry (perp!)



⑦ $\Rightarrow$ Surface has two $\perp$-axes of mirror symmetry near p. $\Longleftrightarrow$ principal directions

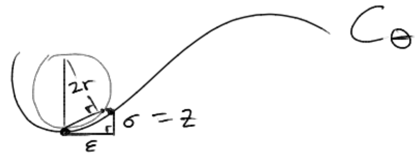
⑧ change of coordinates s.t. x,y axis $\longleftrightarrow$ symmetry directions

(9) under change x, y axis are axes of mirror symmetry so $\begin{matrix} x \mapsto -x \\ y \mapsto -y \end{matrix}$ preserve equation

i.e., $c = 0$ so $z = ax^2 + by^2$

(10) Similar to previous day figure

$$\boxed{\begin{matrix} \dfrac{\varepsilon}{2r} = \dfrac{hyp}{hyp} = \dfrac{base}{\varepsilon} & \dfrac{1}{r} = \dfrac{2\sigma}{\varepsilon^2} \\ \text{then } \varepsilon \approx \varepsilon \implies k = \dfrac{2\sigma}{\varepsilon^2} & \to \dfrac{\varepsilon}{2r} = \dfrac{\sigma}{\varepsilon} \end{matrix}}$$



For $\theta = 0 \leftrightarrow$ x-axis $\quad k_1 = k(0) =$ curvature of $C_\theta \cap xz$ plane $(y = 0)$

we have $z = ax^2$

from before $a = \frac{1}{2} k_1$

$\theta = \frac{\pi}{2} \leftrightarrow$ y-axis $\quad k_2 = k(\frac{\pi}{2})$

... yz plane

$z = by^2 \Rightarrow b = \frac{1}{2} k_2$

$z = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 = \frac{1}{2} k_1 ($

the x, y are orthogonal so

$x = \varepsilon \cos\theta$

$y = \varepsilon \sin\theta$

$$k(\theta) = 2\left(\frac{z}{\varepsilon^2}\right) = 2\left(\frac{\frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2}{\varepsilon^2}\right)$$

$$= \frac{k_1 \cdot \varepsilon^2 \cos^2\theta + k_2 \varepsilon^2 \sin^2\theta}{\varepsilon^2} = k_1 \cos^2\theta + k_2 \sin^2\theta$$

Exercises: Due Monday, Oct. 6

A: Text, Chapter 20, # 5

B: In Desmos, produce a dynamic illustration of the circle of curvature of an ellipse.

1. Parameterize an ellipse
2. Use the formula for curvature of parametric curves we derived in class to get a formula for the radius of curvature of parametric curves.
3. Express the center of the circle of curvature in the the coordinates of your parameterization.